# MODULI OF STABLE BUNDLES ON CURVES

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Abstract. This is a lecture note for a course on moduli of stable bundles on curves. We introduce various tools in moduli theory and apply it to study geometry of moduli of stable bundles on curves. By the end of the course, we prove a special case of a Verlinde formula.

## 1. Introduction to the course with Grassmannian

1.1. **Goal of the course.** Goal of this course is to understand the geometry of moduli space of stable bundles using the standard tools in moduli theory as below.

(1) Geometric invariant theory $\implies$ construction of the moduli space
(2) Deformation theory $\implies$ local properties of the moduli space
(3) Tautological classes and relations $\implies$ global properties of the moduli space
(4) Intersection theory + combinatorics $\implies$ enumerative problems about moduli space

We hope that this course serves as an introduction to the moduli theory in algebraic geometry with a particular focus on moduli of stable bundles on curves.

Moduli space is a parameter space for certain geometric objects. Geometric objects in question come in various flavor. There are moduli space of certain class of abstract varieties (e.g. moduli of stable curves) or Hilbert schemes parametrizing subschemes in a fixed projective variety. On the other hand, one can fix a projective variety and consider moduli space of linear objects on it (e.g. vector bundles or more generally coherent sheaves). It is the latter that this course focuses on.

Moduli space of sheaves serve various purpose in algebraic geometry. First, it is a rich source of new varieties which can also be studied in detail via moduli theoretic techniques. In some cases, classical birational problems for such moduli spaces can be effectively solved via notions in moduli theory, called stability conditions and wall-crossing. Second, there are many interesting interactions with theoretical physics. Verlinde formula computes what's called conformal blocks in two-dimensional conformal field theory. This was later given a mathematical interpretation by means of moduli of stable bundles on curves. In this language, the formula reads

$$\dim H^0(M_C(2,L),\Theta^{\otimes k}) = (k+1)^{g-1}\sum_{j=1}^{2k+1}\frac{(-1)^{j-1}}{\left(\sin\frac{j\pi}{2k+2}\right)^{2g-2}},\quad k\geq 0$$

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in a special case we will consider at the end of the course.[1] Throughout the course, we will learn the necessary ingredients for this formula.

1.2. **Grassmannian.** We illustrate the plan of the course using Grassmannian. Let $V$ be a vector space (over the complex numbers) of rank $n$. Fix an integer $0 < k < n$. Define a Grassmannian

$$\mathrm{Gr}(V,k) := \big\{(W,\phi)\big\}/\sim$$

as a set of pairs where

(1) $W$ is a vector space of rank $k$,
(2) $\phi : V \twoheadrightarrow W$ is a surjective linear map,

up to an equivalence relation $\sim$ for a commutative diagram

$$\begin{tikzcd} V \arrow[r, two heads, "\phi_1"] \arrow[d, equal] & W_1 \arrow[d, "\psi"] \\ V \arrow[r, two heads, "\phi_2"'] & W_2. \end{tikzcd}$$

One can equivalently define Grassmannian using a subspace $K := \ker(\phi) \subset V$. However, definition using a quotient will be crucial when we encounter quot schemes later.

1.2.1. *Construction.* We can give $\mathrm{Gr}(V,k)$ a structure of algebraic variety. Choose a basis $\{v_1, \cdots, v_n\}$ of $V$, hence identifying $V \simeq \mathbb{C}^n$. Then the above data amounts to a full rank $k \times n$ matrix $A = (a_{ij})$ up to row operations. For each subset $J \subset [n]$ of size $k$, denote the corresponding $k \times k$ minor matrix by $A_J$. Let $U_J$ be a subset of $\mathrm{Gr}(\mathbb{C}^n, k)$ consisting of a matrix $A$ whose minor $A_J$ is invertible. Then we have a bijection

$$f_J : U_J \xrightarrow{\sim} \mathbb{C}^{k(n-k)},$$

where the map reads entries of the matrix $A \mapsto A \cdot (A_J)^{-1}$ omitting $j$'th columns for $j \in J$. Since $\mathrm{Gr}(\mathbb{C}^n, k)$ is a union of $U_J$'s, this provides a chart for the Grassmannian. In the overlap of the charts, the change of coordinate is given by a rational function. This is because determiant is polynomial in the entries and there is a formula for the inverse of a matrix

$$M^{-1} = \frac{1}{\det(M)} \cdot \mathrm{adj}(M).$$

Projectivity can also be shown via Plucker embedding [2]

$$i : \mathrm{Gr}(V,k) \hookrightarrow \mathbb{P}(\Lambda^k V), \quad [V \twoheadrightarrow W] \mapsto [\Lambda^k V \twoheadrightarrow \Lambda^k W].$$

Therefore, $\mathrm{Gr}(V,k)$ is given a structure of smooth projective variety of dimension $k(n-k)$.

As a moduli space, $\mathrm{Gr}(V,k)$ comes with a universal object $(\mathcal{W}, \Phi)$ where

[1] In the formula, $C$ is a smooth projective curve of genus $g \geq 2$, $L$ is a line bundle of odd degree and $M_C(2,L)$ is a moduli of stable bundles of rank 2 and determinant $L$. Determinant line bundle $\Theta$ is the ample generator of the picard group of $M_C(2,L)$.

[2] According to the identification $V \simeq \mathbb{C}^n$, this maps $A$ to a collection of $\det(A_J)$ for each $J$.

(1) $\mathcal{W}$ is a vector *bundle* of rank $k$ on $\mathrm{Gr}(V,k)$,
(2) $\Phi : V \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Gr}(V,k)} \twoheadrightarrow \mathcal{W}$ is a surjective map between vector bundles.

This is universal in sense that at each point the universal object restricts to a quotient that the point represents. Precisely, if $p \in \mathrm{Gr}(V,k)$ represents a quotient $[V \xrightarrow{\phi} W]$, we have

$$(\mathcal{W},\Phi)\Big|_p = (W,\phi).$$

Construction of the universal quotient can also be done by the chart argument as above.

1.2.2. *Local property.* Since $\mathrm{Gr}(V,k)$ is constructed via gluing affine space of dimension $k(n-k)$, there is no further local properties to be known. However, we can still ask for a better understanding of a tangent space $T_p$ of a Grassmannian at a point $p$. Precisely, we want to identify a tangent space $T_p$ using a quotient data $(W,\phi)$ that $p$ represents. Recall that $\mathrm{Gr}(V,k)$ is of dimension $k(n-k)$. On the other hand, $W$ and $K := \ker(V \xrightarrow{\phi} W)$ are of rank $k$ and $n-k$, respectively. This suggests that $T_p$ is given by a tensor product of $W$ and $K$ with possibly some dualized. Of course, $T_p$ and $K^{\otimes\pm} \otimes W^{\otimes\pm}$ are all isomorphic to each other as a vector space since they have an equal dimension. However, we want a *natural* isomorphism such that it glues globally.

We show that the natural one is given by

$$T_p \simeq \mathrm{Hom}(K,W) = K^\vee \otimes W.$$

Recall that Zariski tangent space $T_p$ is identified with a set of morphisms

$$\mathrm{Mor}_p(\mathrm{Spec}\,\mathbb{C}[\epsilon]/\epsilon^2, \mathrm{Gr}(V,k)).$$

This corresponds to[3] a set of pairs $(\mathcal{W},\Phi)$ where

(1) $\mathcal{W}$ is a *flat* $\mathbb{C}[\epsilon]/\epsilon^2$-module of rank $k$,
(2) $\Phi : V \otimes \mathbb{C}[\epsilon]/\epsilon^2 \twoheadrightarrow \mathcal{W}$ is a surjective $\mathbb{C}[\epsilon]/\epsilon^2$-linear map,
(3) $(\mathcal{W},\Phi)\Big|_{\epsilon=0} = (W,\phi)$.

By tensoring $0 \to \mathbb{C} \xrightarrow{\cdot\epsilon} \mathbb{C}[\epsilon]/\epsilon^2 \to \mathbb{C} \to 0$ to the exact sequence $0 \to \mathcal{K} \to V \otimes \mathbb{C}[\epsilon]/\epsilon^2 \to \mathcal{W} \to 0$, we obtain a diagram

[3] This type of correspondence follows from the universal property of Grassmannian as a moduli space. We will study universal property of moduli spaces later with concepts of moduli functor and representability.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & V \oplus \epsilon \cdot V & \longrightarrow & \mathcal{W} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the exactness on the right column follows from flatness of $\mathcal{W}$ hence the exactness of the left column. From this data which correspond to the tangent vector in $T_p$, we associate an element $f \in \mathrm{Hom}_{\mathbb{C}}(K, W)$ as follows. For each $k \in K$, choose a lift to $\mathcal{K}$ and denote it as $k + \epsilon \cdot v$. Then $v$ is well-defined up to ambiguity of $K$ hence define an element $f(k) \in V/K = W$. (Check: $f$ is $\mathbb{C}$-linear map). Conversely, if we are given $f \in \mathrm{Hom}(K, W)$, we can define

$$\mathcal{K} := \left\{ a + \epsilon \cdot b \,\middle|\, a \in K, \quad b + K = f(a) \right\} \subseteq V \oplus \epsilon V \twoheadrightarrow \mathcal{W}.$$

This provides a natural bijection between $T_p$ and $\mathrm{Hom}_{\mathbb{C}}(K, W)$ proving the claim.

This natural identification yields a description of a tangent *bundle* in terms of the universal data. Therefore, we obtain a formula for the tangent bundle

$$T_{\mathrm{Gr}(V,k)} \simeq \mathcal{H}\mathrm{om}(\mathcal{K}, \mathcal{W}) = \mathcal{K}^\vee \otimes \mathcal{W},$$

where $0 \to \mathcal{K} \to V \otimes \mathcal{O}_{\mathrm{Gr}(V,k)} \to \mathcal{W} \to 0$ is the universal quotient. Somewhat ironically, local study of a moduli space yielded an important global identification of the tangent space using the universal object.

1.2.3. *Global property.* Now we turn our attention to global properties of a Grassmannian $\mathrm{Gr}(V, k)$, namely the cohomology ring. In general, cohomology of a moduli space can be quite complicated. However, there are some natural (in other word, geometric) classes that arise from the universal object, called tautological classes. In the case of Grassmannian, we use the universal quotient

$$0 \to \mathcal{K} \to V \otimes \mathcal{O}_{\mathrm{Gr}(V,k)} \to \mathcal{W} \to 0.$$

Tautological classes are defined to be polynomials in classes

$$c_i(\mathcal{K}),\ c_i(\mathcal{W}) \in H^{2i}(\mathrm{Gr}(V,k), \mathbb{Z}).$$

By a cohomology group on the right hand side, we first take analytification of an algebraic variety and then take singular cohomology. Recall that chern classes are defined for any complex vector bundles on topological spaces with mild assumptions which are satisfied for analytification of any variety over $\mathbb{C}$.

Note that it is enough to work with $c_i := c_i(\mathcal{W})$'s only due to Whitney sum formula for the total Chern class

$$c(\mathcal{K}) \cdot c(\mathcal{W}) = c(V \otimes \mathcal{O}_{\mathrm{Gr}(V,k)}) = 1.$$

More concretely, we have

$$c_i(\mathcal{K}) = \left[\frac{1}{1 + c_1 + \cdots + c_k}\right]_i$$

where $[\ ]_i$ pick outs class of cohomological degree $2i$. We show that cohomology ring of $\mathrm{Gr}(V,k)$ with $\mathbb{Q}$-coefficient[4] is generated by tautological classes $c_1, \ldots, c_k$. In other words, we show that there is a surjection between super-graded $\mathbb{Q}$-algebras

$$\mathbb{Q}[c_1, \ldots, c_k] \twoheadrightarrow H^*(\mathrm{Gr}(V,k), \mathbb{Q}).$$

We prove this using a diagonal technique. Consider a diagonal map

$$\Delta : \mathrm{Gr}(V,k) \hookrightarrow \mathrm{Gr}(V,k) \times \mathrm{Gr}(V,k).$$

If we have a Kunneth decomposition

$$\Delta_* 1 = \sum_{i \in I} \gamma_i^L \otimes \gamma_i^R \in H^*(\mathrm{Gr}(V,k), \mathbb{Q}) \otimes H^*(\mathrm{Gr}(V,k), \mathbb{Q}),$$

then $\{\gamma_i^L\}_{i \in I}$ $\mathbb{Q}$-linearly spans cohomology ring over $\mathbb{Q}$. Therefore, it suffices to find a Kunneth decomposition of a diagonal with all $\gamma_i^L$ being tautological classes. Denote by

$$0 \to \mathcal{K}_i \to V \otimes \mathcal{O}_{\mathrm{Gr} \times \mathrm{Gr}} \to \mathcal{W}_i \to 0, \quad i = 1, 2$$

a pull back of the universal quotient along the projection map $\pi_i : \mathrm{Gr} \times \mathrm{Gr} \to \mathrm{Gr}$ to the $i$'th factor. Then we can consider a morphism defined as a composition

$$s : \mathcal{K}_2 \to V \otimes \mathcal{O}_{\mathrm{Gr} \times \mathrm{Gr}} \to \mathcal{W}_1.$$

Note that a zero set of this morphism $Z(s) \subseteq \mathrm{Gr} \times \mathrm{Gr}$ is exactly the diagonal. In other word, we have a section $s$ of a vector bundle $\mathcal{K}_2^\vee \otimes \mathcal{W}_1$ that cuts out the diagonal canonically in the expected dimension. This implies the formula for the diagonal

$$\Delta_* 1 = c_{k(n-k)}(\mathcal{K}_2^\vee \otimes \mathcal{W}_1).$$

By Chern class formula for a tensor product, it is clear that we have a Kunneth decomposition of the diagonal with $\gamma_i^L$ being polynomials in $c_1, \ldots, c_k$.

[4]The same statement is actually true for integral cohomology ring.

Once we find generators of cohomology ring, it is natural to ask relations between them. There are also naturally or geometrically given relations. This follows from the fact that chern classes vanish beyond the rank of a vector bundle. Therefore, we have

$$R_i := \left[\frac{1}{1+c_1+\cdots+c_k}\right]_i = 0, \quad n-k < i \le k(n-k).$$

In other words, the previously defined surjection factors through a quotient algebra

$$\mathbb{Q}[c_1,\ldots,c_k]/(R_{n-k+1},\cdots,R_{k(n-k)}) \twoheadrightarrow H^*(\mathrm{Gr}(V,k),\mathbb{Q}).$$

It is a non-trivial fact that the above morphism is an isomorphism even with $\mathbb{Z}$-coefficient. In principal, this can be done by computing the dimension of the finite dimensional[5] $\mathbb{Q}$-algebra on the left hand side and match it with a topological Euler characteristic $\chi^{\mathrm{top}}(\mathrm{Gr}(V,k)) = \binom{n}{k}$. We show the isomorphism in a special case of $\mathrm{Gr}(\mathbb{C}^4,2)$ following this strategy later.

1.2.4. *Intersection theory.* Intersection theory for Grassmannian, known as Schubert calculus, is a well-studied subject. In this section, we study a particular question that does not require a full stretch with Schubert calculus.

Consider an exponential exact sequence (in analytic category)

$$0 \to 2\pi i\cdot\mathbb{Z} \to \mathcal{O}_{\mathrm{Gr}(V,k)} \xrightarrow{\exp} \mathcal{O}^*_{\mathrm{Gr}(V,k)} \to 0.$$

This induces an injection

$$\mathsf{Pic}(\mathrm{Gr}(V,k)) \to H^2(\mathrm{Gr}(V,k),\mathbb{Z})$$

because of the vanishing $H^1(\mathrm{Gr}(V,k),\mathbb{Z}) = 0$. Recall that $H^2(\mathrm{Gr}(V,k),\mathbb{Q})$ is $\mathbb{Q}$-spanned over a rational number by a tautological class $c_1 = c_1(\mathcal{W})$. This proves that

$$\mathsf{Pic}(\mathrm{Gr}(V,k))_{\mathbb{Q}} \simeq \mathbb{Q}\cdot\mathcal{L},$$

where $\mathcal{L} := \det(\mathcal{W})$ is the ample line bundle giving the Plucker embedding. In fact, the same statement is grue over the integral coefficient.

As an analogues of the Verlinde formula for the Grassmannian, we may ask

$$\dim H^0(\mathrm{Gr}(V,k),\mathcal{L}^{\otimes m}) = \ ?$$

for $m \ge 0$. From the tangent bundle formula $T_{\mathrm{Gr}} = \mathcal{K}^\vee \otimes \mathcal{W}$, we deduce that the canonical bundle is given by

$$K_{\mathrm{Gr}} = \Big(\det(\mathcal{K}^\vee)^{\otimes k} \otimes \det(\mathcal{W})^{\otimes n-k}\Big)^{-1} = \mathcal{L}^{\otimes -n}.$$

In particular, $\mathrm{Gr}(V,k)$ is a Fano variety of index $n = \dim(V)$. From Kodaira vanishing, we know that

$$H^i(\mathrm{Gr}(V,k),\mathcal{L}^{\otimes m}) = 0, \quad i>0,\ m\ge 0,$$

[5]It is not trivial that this is even a finite dimensional $\mathbb{Q}$-vector space.

hence it suffices to compute the holomorphic Euler characteristic $\chi(\mathrm{Gr}(V,k),\mathcal{L}^{\otimes m})$.

The holomorphic Euler characteristics of a line bundle can be computed by means of *intersection theory*. More precisely, Hirzebruch-Riemann-Roch formula says

$$\chi(\mathrm{Gr}(V,k),\mathcal{L}^{\otimes m}) = \int_{\mathrm{Gr}(V,k)} e^{m\cdot c_1(\mathcal{L})}\cdot \mathrm{td}(T_{\mathrm{Gr}(V,k)}).$$

We will explain necessary ingredients to understand this formula in the next section. The upshot is that the integrand can be naturally expressed using the tautological classes. In the end, we only need to determine the integration homomorphism

$$\int_{\mathrm{Gr}(V,k)} : \mathbb{Q}[c_1,\dots,c_k]/(R_{n-k+1},\cdots,R_{k(n-k)}) \to \mathbb{Q}.$$

We will see the computation for a special case of $\mathrm{Gr}(\mathbb{C}^4,2)$ in the next section.

## 2. Fundamentals of sheaf theory

In this section, we review some of the fundamentals of sheaf theory in algebraic geometry.

### 2.1. **Coherent sheaves.**

Let $X$ be a *scheme* (always over $\mathbb{C}$ otherwise specified). This is a locally ringed space $(X,\mathcal{O}_X)$ such that there exists open covering $\{U_\alpha\}$ with $(U_\alpha,\mathcal{O}_{U_\alpha})\simeq \mathrm{Spec}A_\alpha$ for some $\mathbb{C}$-algebra $A_\alpha$. By a *sheaf* $F$ on $X$, we always mean $\mathcal{O}_X$-module. This is a data of

$$F(U) : \Gamma(U,\mathcal{O}_X) - \text{module}$$

for every open set $U$ that are compatible with respect to restriction maps. A sheaf $F$ on a locally notherian scheme $X$ is *coherent* if for every affine open subset $U=\mathrm{Spec}A\subset X$, we have

$$F\Big|_U \simeq \widetilde{M}$$

for some finitely generated $A$-module $M$. Here $\widetilde{M}$ denotes a standard construction of quasi-coherent sheaf on a affine scheme.

Coherent sheaves are a fundamental object in algebraic geometry. There are various examples some of which we list below. We assume that $X$ is locally of finite type over $\mathbb{C}$.

(1) Structure sheaf $\mathcal{O}_X$
(2) Vector bundle of finite rank = locally free sheaf of finite rank
(3) Kahler differential $\Omega_X$
(4) Skyscraper sheaf $k(x)$ where $x\in|X|$ [6]
(5) Ideal sheaf $I_{Z/X}$ for a closed subscheme $i : Z\hookrightarrow X$
(6) For a proper morphism $f : X\to Y$ and a coherent sheaf $F$ of $X$, a pushforward $f_*F$.

[6] We denote the set of closed point of a scheme by $|X|$.

Coherent sheaves can be thought of as a generalization of vector bundles. Unlike vector bundles, coherent sheaves have an advantage of being closed under kernel and cokernel. In other words, coherent sheaves form an abelian category $\mathrm{Coh}(X)$. For a coherent sheaf $F$, we associate a scheme theoretic support $\mathrm{supp}(F)$ via annihilating ideal

$$\ker(\mathcal{O}_X \to \mathcal{H}\mathrm{om}(F, F)).$$

Intuitively, one can think of a coherent sheaf $F$ as a "vector bundle" on $\mathrm{supp}(F)$ with some singularities.

Sheaf cohomology is a crucial tool for the study of coherent sheaf. Let $X$ be a proper scheme and $F \in \mathrm{Coh}(X)$. Sheaf cohomology is a collection of finite dimensional $\mathbb{C}$-vector spaces $H^i(X, F)$ for $i \geq 0$. They satisfies various properties some of which we record below.

(1) $H^0(X, F) = \Gamma(X, F)$.
(2) For any $\phi : F \to G$, we have $H^i(f) : H^i(X, F) \to H^i(X, G)$.
(3) For any short exact sequence $0 \to F \to G \to H \to 0$, we have a long exact sequence

$$\cdots \to H^i(X, F) \to H^i(X, G) \to H^i(X, H) \to H^{i+1}(X, F) \to \cdots .$$

(4) $H^i(X, F) = 0$ if $i > \dim(\mathrm{supp}(F))$.

More generally, we have ext groups $\mathrm{Ext}^i_X(F, G)$ for $i \geq 0$ with analogous properties. Cohomology groups are recovered from $\mathrm{Ext}^i(\mathcal{O}_X, F) = H^i(X, F)$.

2.2. **Serre duality.** Assume that $X$ is a smooth projective connected scheme over $\mathbb{C}$ of dimension $d$. For such $X$, sheaf cohomologies $H^i(X, F)$ admits two important structures; Serre duality and index theorem. We study Serre duality here.

Recall the definition of a canonical line bundle

$$K_X := \det(\Omega_X) = \Lambda^d(\Omega_X).$$

Serre duality is a collection of *natural* isomorphisms

$$\mathrm{Ext}^i(F, G) \simeq \mathrm{Ext}^{d-i}(G, F \otimes K_X)^\vee$$

for each $i$ and $F, G \in \mathrm{Coh}(X)$. Naturality here means a functoriality in both $F$ and $G$. This isomorphism can be factorized as

$$\mathrm{Ext}^i(F, G) \otimes \mathrm{Ext}^{d-i}(G, F \otimes K_X) \xrightarrow{\cup} \mathrm{Ext}^d(F, F \otimes K_X) \xrightarrow{\mathrm{tr}} H^d(X, K_X) \xrightarrow{\int_X} \mathbb{C}.$$

One can think of Serre duality as a "generalization" of Poincare duality in some sense. Denote a sheaf of algebraic $p$-forms by $\Omega^p := \Lambda^p(\Omega_X)$. We have a perfect paring

$$\Omega^p \otimes \Omega^{d-p} \xrightarrow{\wedge} K_X$$

that induces an isomorphism $\Omega^{d-p} \simeq (\Omega^p)^\vee \otimes K_X$. Therefore, Serre duality implies an isomorphism

$$H^q(X, \Omega^p) \simeq H^{d-q}(X, \Omega^{d-p})^\vee.$$

On the other hand, Hodge decomposition says

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^q(X, \Omega^p).$$

Then Poincare duality $H^i(X, \mathbb{C}) \simeq H^{2d-i}(X, \mathbb{C})$ is refined on each summand by Serre duality. However, Serre duality is a far reaching generalization that works for arbitrary Ext groups between any coherent sheaves.

2.3. **Index theorem.** Define the holomorphic Euler characteristic as

$$\chi(X, F) := \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X, F).$$

While individual cohomology groups $H^i(X, F)$ are difficult to study, index type theorem tells us how to compute the holomorphic Euler characteristics in terms of *topological* data. This is surprising because each term $\dim_{\mathbb{C}} H^i(X, F)$ are not topological invariant and sensitive to algebraic structure. For example, if $C$ is an elliptic curve then

$$H^0(C, \mathcal{O}_C(p-q)) = \begin{cases} 0\,, & \text{if } p \neq q, \\ \mathbb{C}, & \text{otherwise.} \end{cases}$$

2.3.1. *Chern classes.* Recall that chern classes were defined for vector bundles. For their generalization to coherent sheaves, we introduce Grothendieck's $K$-group. Algebraic $K$-theory $K^0(X)$ is defined as a quotient of a free group generated by symbols $[V]$ for each vector bundle $V$ up to a relation $[V_2] = [V_1] + [V_3]$ whenever there is a short exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0.$$

This is in fact a ring with respect to a tensor product. We have a well-defined map

$$\mathrm{Coh}(X) \to K^0(X), \quad F \mapsto \sum_i (-1)^i [V_i]$$

where $V_\bullet \to F \to 0$ is a finite resolution of a coherent sheaf.[7] On the other hand, we have a well-defined total Chern class map

$$c(-) : K^0(X) \to H^*(X, \mathbb{Z}), \quad [V] \mapsto c(V)$$

thanks to Whitney sum formula. Therefore we can define chern classes for coherent sheaves using a composition

$$c(-) : \mathrm{Coh}(X) \to K^0(X) \to H^*(X, \mathbb{Z}).$$

[7]Such a resolution always exists because $X$ is smooth projective.

2.3.2. *Splitting principle.* We need other characteristic classes to state the index theorem. For this, we recall splitting principle. Let $V$ be a vector bundle of rank $n$. Pretend for now that $V$ splits into line bundles $V = L_1 \oplus \cdots \oplus L_n$. Denote $x_i := c_1(L_i)$, which we call Chern roots. By Whitney sum formula, we have

$$1 + c_1(V) + \cdots + c_n(V) = (1 + x_1) \cdots (1 + x_n)$$

or equivalently

$$c_i(V) = \sigma_i(x_1, \ldots, x_n)$$

where $\sigma_i$ is the $i$'th elementary symmetric polynomial. Using these Chern roots, we define new characteristic classes. First, we define chern character as

$$\mathrm{ch}(V) := \sum_{i=1}^{n} e^{x_i} \in H^*(X, \mathbb{Q}).$$

The upshot is that even though that $V$ may not splits, we can make sense of the definition by a symmetric reduction. More precisely, we have

$$\mathrm{ch}_k(V) = \sum_{i=1}^{n} \frac{x_i^k}{k!} \in \mathbb{Q}[x_1, \ldots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \ldots, \sigma_n],$$

hence $\mathrm{ch}_k(V)$ is defined by setting $\sigma_i$ to be $c_i(V)$. By a same method, we define todd class

$$\mathrm{td}(V) := \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} \in H^*(X, \mathbb{Q}).$$

Even though we defined chern character and todd class only for vector bundles, they factor through a $K$-group, hence is also defined for coherent sheaves.

There is an useful fact about the first non-trivial Chern character of a coherent sheaf. Let $F$ be a nonzero coherent sheaf on a smooth projective scheme $X$. Suppose that $\dim(\mathrm{supp}F) = k$. Then we have a vanishing of a Chern character $\mathrm{ch}_i(F) = 0$ for $i < k$. And the first non-trivial chern character $\mathrm{ch}_k(F)$ is given by an positive integral combination of the $k$-cycles supported on $\mathrm{supp}F$ with coefficients determined by a length of $F$ on such each cycle. For example, let $F := i_*V$ for a smooth irreducible subscheme $i : Z \hookrightarrow X$ of dimension $k$ and a vector bundle $V$ of rank $r$ on $Z$. Then we have $\mathrm{ch}_i(F) = 0$ for $i < k$ and $\mathrm{ch}_k(V) = r[Z]$.

2.3.3. *Hirzebruch-Riemann-Roch.* Now we are ready to state the index theorem, called Hirzebruch-Riemann-Roch formula. For any coherent sheaf $F$ on a smooth projective scheme $X$, we have

$$\chi(X, F) = \int_{[X]} \mathrm{ch}(F) \cdot \mathrm{td}(T_X).$$

More generally, if we define

$$\chi(F, G) := \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i(F, G),$$

we have

$$\chi(F,G)=\int_{[X]}\mathrm{ch}^{\vee}(F)\cdot\mathrm{ch}(G)\cdot\mathrm{td}(T_X)$$

where dual chern character is defined as $\mathrm{ch}^{\vee}(F):=\sum_k(-1)^k\mathrm{ch}_k(F)$.

**Example 1.** We illustrate how to use Hirzebruch-Riemann-Roch formula in an example. Let $X=\mathbb{P}^n$ and consider $F=\mathcal{O}_{\mathbb{P}^n}(k)$. By Hirzebruch-Riemann-Roch formula, we have

$$\chi(\mathbb{P}^n,\mathcal{O}(k))=\int_{\mathbb{P}^n}\mathrm{ch}(\mathcal{O}(k))\cdot\mathrm{td}(T_{\mathbb{P}^n}).$$

By definition of chern character, $\mathrm{ch}(\mathcal{O}(k))=e^{kH}$ where $H:=c_1(\mathcal{O}(1))$ is the hyperplane class. On the other hand, we have an Euler exact sequence

$$0\to\mathcal{O}\to\mathcal{O}(1)\otimes\mathbb{C}^{n+1}\to T_{\mathbb{P}^n}\to 0.$$

From multiplicativity of the todd class, we obtain

$$\mathrm{td}(T_{\mathbb{P}^n})=\mathrm{td}(\mathcal{O}(1))^{n+1}\cdot\mathrm{td}(\mathcal{O})=\left(\frac{H}{1-e^{-H}}\right)^{n+1}.$$

We compute the integral using the residue formula as below:

$$\begin{aligned}\chi(\mathbb{P}^n,\mathcal{O}(k))&=\int_{\mathbb{P}^n}e^{kH}\cdot\left(\frac{H}{1-e^{-H}}\right)^{n+1}\\&=\mathrm{Res}_{H=0}\frac{e^{kH}}{(1-e^{-H})^{n+1}}dH\\&=\mathrm{Res}_{x=1}\frac{x^k}{(1-x^{-1})^{n+1}}\frac{dx}{x}\\&=\mathrm{Res}_{x=1}\frac{x^{k+n}}{(x-1)^{n+1}}dx\\&=\mathrm{Res}_{y=0}\frac{(y+1)^{k+n}}{y^{n+1}}dy\\&=\binom{k+n}{n}.\end{aligned}$$

The result is compatible with a fact that for each $k\geq 0$,

$$H^0(\mathbb{P}^n,\mathcal{O}(k))\simeq\Big(\mathbb{C}[x_0,\cdots,x_n]\Big)_k\quad\text{and}\quad H^i(\mathbb{P}^n,\mathcal{O}(k))=0,\ i>0.$$

Here $\Big(-\Big)_k$ takes homogeneous part of degree $k$.

**Example 2.** We apply Hirzebruch-Riemann-Roch for "Verlinde number" of $\mathrm{Gr}(\mathbb{C}^4,2)$. Recall that we have a surjection

$$\mathbb{Q}[c_1,c_2]/(R_3,R_4)\twoheadrightarrow H^*(\mathrm{Gr}(\mathbb{C}^4,2),\mathbb{Q})$$

where

$$R_3=2c_1c_2-c_3^3,\quad R_4=c_1^4-3c_1^2c_2+c_2^2.$$

We first show that this is in fact an isomorphism by computing the dimension of of each graded pieces, called Hilbert series. For that, we use a locally free resolution of finite length over a graded ring $A := \mathbb{Q}[c_1, c_2]$:

$$0 \to A(-7) \xrightarrow{\binom{R_4}{-R_3}} A(-3) \oplus A(-4) \xrightarrow{(R_3\ \ R_4)} A \to \mathbb{Q}[c_1, c_2]/(R_3, R_4) \to 0.$$

Therefore Hilbert series of $\mathbb{Q}[c_1, c_2]/(R_3, R_4)$ is computed as

$$(1 - t^3 - t^4 + t^7) \cdot \mathsf{Hilb}(A) = \frac{1 - t^3 - t^4 + t^7}{(1-t)(1-t^2)} = (1 + t + t^2)(1 + t^2).$$

By setting $t = 1$, we obtain 6 which is equal to a topological Euler characteristic of $\mathrm{Gr}(\mathbb{C}^4, 2)$. This proves the desired isomorphism. By Hirzebruch-Riemann-Roch formula, we have

$$\begin{aligned}\chi(\mathrm{Gr}(\mathbb{C}^4, 2), \mathcal{L}^{\otimes m}) &= \int_{\mathrm{Gr}(\mathbb{C}^4,2)} e^{m \cdot c_1(\mathcal{L})} \cdot \mathrm{td}(T_{\mathrm{Gr}}) \\ &= \int_{\mathrm{Gr}(\mathbb{C}^4,2)} e^{m \cdot c_1(\mathcal{L}) + \frac{1}{2} c_1(T_{\mathrm{Gr}})} \cdot \widehat{A}(T_{\mathrm{Gr}}).\end{aligned}$$

The second equality is coming from the modification of a characteristic class using the identity

$$\mathrm{td}(x) = \frac{x}{1 - e^{-x}} = e^{x/2} \cdot \frac{x/2}{(e^{x/2} - e^{-x/2})/2} = e^{x/2} \cdot \frac{x/2}{\sinh(x/2)} = e^{x/2} \cdot \widehat{A}(x).$$

We express the integrand using the tautological classes. Clearly, $c_1(\mathcal{L}) = c_1$ and $c_1(T_{\mathrm{Gr}}) = 4c_1$. By definition of $\widehat{A}$-genus the first few terms look like

$$\widehat{A}(T_{\mathrm{Gr}}) = 1 - \frac{1}{24} p_1(T_{\mathrm{Gr}}) + \frac{1}{5760}\big(7p_1(T_{\mathrm{Gr}})^2 - 4p_2(T_{\mathrm{Gr}})\big) + \cdots$$

where we define Pontryagin classes[8] as

$$p_i(E) := (-1)^i\, c_{2i}(E \oplus E^\vee).$$

From computation, we find

$$\begin{aligned}1 - p_1(T_{\mathrm{Gr}}) + p_2(T_{\mathrm{Gr}}) &= c(T_{\mathrm{Gr}} \oplus T_{\mathrm{Gr}}^\vee) \\ &= c(\mathcal{K}^\vee \otimes \mathcal{W} \oplus \mathcal{K} \otimes \mathcal{W}^\vee) \\ &= c\Big(\mathcal{W}^{\oplus 4} \oplus (\mathcal{W}^\vee)^{\oplus 4} - (\mathcal{W}^\vee \otimes \mathcal{W})^{\oplus 2}\Big) \\ &= \frac{(1 + c_1 + c_2)^4 (1 - c_1 + c_2)^4}{(1 - c_1^2 + 4c_2)^2}.\end{aligned}$$

Summing up everything, we have expressed "Verlinde number"

$$\chi(\mathrm{Gr}(\mathbb{C}^4, 2), \mathcal{L}^{\otimes m}) = \int_{\mathrm{Gr}(\mathbb{C}^4,2)} e^{(m+2)c_1} \cdot \left(1 - \frac{1}{24} p_1(T_{\mathrm{Gr}}) + \frac{1}{5760}\big(7p_1(T_{\mathrm{Gr}})^2 - 4p_2(T_{\mathrm{Gr}})\big)\right)$$

[8]Pontryagin classes only depend on the $\mathbb{R}$-vector bundle structure.

as an integral of explicit tautological class. We leave the computation of the integral using the relations $R_3 = R_4 = 0$ as an exercise. The only non-trivial step is to determine the integration map

$$\int_{\mathrm{Gr}(\mathbb{C}^4,2)} : \mathbb{Q}[c_1, c_2]/(R_3, R_4) \to \mathbb{Q}$$

in terms of the tautological classes. This can be done in several ways. We can use either the identity

$$\chi(\mathrm{Gr}(\mathbb{C}^4, 2), \mathcal{O}_{\mathrm{Gr}}) = 1,$$

or degree of the Plucker embedding

$$\int_{\mathrm{Gr}(\mathbb{C}^4,2)} c_1^4 = 2.$$

We can find "Verlinde numbers" of $\mathrm{Gr}(\mathbb{C}^4, 2)$ using $K$-theoretic intersection theory too. Using the same argument as in the cohomological case, we can prove that there is an isomorphism

$$\mathbb{Q}[\Lambda^1, \Lambda^2]/(R_3, R_4) \twoheadrightarrow K^*(\mathrm{Gr})_{\mathbb{Q}}, \quad 1 \mapsto \mathcal{O}_{\mathrm{Gr}},\ \Lambda^i \mapsto \Lambda^i(\mathcal{W}).$$

Here $K$-theoretic tautological relations are given by

$$R_i := [t^i]\,\frac{(1+t)^4}{1 + t\Lambda^1 + t^2\Lambda^2}, \quad i = 3,\, 4.$$

By definition, Verlinde number is defined as $\chi(\mathrm{Gr}, (\Lambda^2)^{\otimes m})$. To find this, we use the exact sequence

$$0 \to (\Lambda^2)^\vee \to \mathcal{O}_{\mathrm{Gr}} \to \mathcal{O}_H \to 0$$

where $H$ denotes the hyperplane section of the Plucker embedding. This induces a relation in $K$-theory

$$\mathcal{O}_H = 1 - (\Lambda^2)^\vee.$$

Therefore we can rewrite the desired integrand as

$$\begin{aligned}
(\Lambda^2)^{\otimes m} &= \Big(1 - \mathcal{O}_H\Big)^{\otimes -m} \\
&= \sum_{k=0}^{4} \binom{-m}{k} \big(\mathcal{O}_H\big)^{\otimes k} \\
&= \sum_{k=0}^{4} (-1)^k \binom{m+k-1}{k} \big(\mathcal{O}_H\big)^{\otimes k},
\end{aligned}$$

where we used the fact that $\mathcal{O}_H$ is nilpotent with $\big(\mathcal{O}_H\big)^{\otimes 5} = 0$. By linearity,

$$\chi(\mathrm{Gr}, (\Lambda^2)^{\otimes m}) = \sum_{k=0}^{4} (-1)^k \binom{m+k-1}{k} \chi(\mathrm{Gr}, \big(\mathcal{O}_H\big)^{\otimes k}).$$

On the other hand, we can evaluate all the $\chi$'s on the right hand side as

$$\chi(\mathrm{Gr}, \mathcal{O}_{H_1 \cap \cdots \cap H_i}) = 1, \quad i = 0, 1, 2, 3, \quad \chi(\mathrm{Gr}, \mathcal{O}_{H_1 \cap \cdots \cap H_4}) = 2.$$

This is because $H_1 \cap \cdots \cap H_i$'s are smooth conics except that $H_1 \cap \cdots \cap H_4$ is two points. So we conclude that

$$\chi(\mathrm{Gr}, \mathcal{L}^{\otimes m}) = 1 - \frac{m}{1!} + \frac{m(m+1)}{2!} - \frac{m(m+1)(m+2)}{3!} + 2 \cdot \frac{m(m+1)(m+2)(m+3)}{4!}.$$

You may compare the answers given by cohomological intersection theory and $K$-theoretic intersection theory.

2.3.4. *Grothendieck-Riemann-Roch.* Let $f : X \to S$ be a morphism between smooth projective varieties. There is a well-defined pushforward operation in $K$-theory

$$Rf_* : K^0(X) \to K^0(S).$$

For vector bundle $V$ on $X$, this is defined as

$$Rf_* V := \sum_{i \geq 0} (-1)^i [R^i f_* V] \in K^0(S).$$

This definition factors through $K$-group because of the long exact sequence between higher direct image sheaves.

Suppose only for this paragraph that $f$ is a smooth morphism and we are given a vector bundle $V$ on $X$. Since restricted vector bundles $V_s \to X_s$ vary continuously, their topological data stays locally constant. Therefore $\chi(X_s, V_s)$ computes the rank of a $K$-theory class $Rf_* V \in K^0(S)$. Since rank of a $K$-theory class is the zeroth chern character $\mathrm{ch}_0(Rf_* V) \in H^0(S, \mathbb{Q})$, we may ask what are the rest of the chern characters. Grothendieck-Riemann-Roch is the formula is what computes the total chern character of $Rf_* V$, hence providing a generalization of Hirzebruch-Riemann-Roch in the relative setting.

**Theorem 3.** *Let $f : X \to S$ be a morphism between smooth projective varieties. Then the following diagram commutes:*

$$\begin{tikzcd} K^0(X) \arrow[r, "Rf_*"] \arrow[d, "\mathrm{ch}(-).\mathrm{td}(X)"'] & K^0(S) \arrow[d, "\mathrm{ch}(-).\mathrm{td}(S)"] \\ H^*(X, \mathbb{Q}) \arrow[r, "f_*"'] & H^*(S, \mathbb{Q}). \end{tikzcd}$$

*In other words, we have*

$$\mathrm{ch}(Rf_*(F)).\mathrm{td}(S) = f_*\big(\mathrm{ch}(F).\mathrm{td}(X)\big).$$

Grothendieck-Riemann-Roch is a powerful tool in moduli theory. In this lecture, we will apply the formula in the case where the morphism is a projection to a moduli space

$$p : M \times X \to M$$

and the sheaf $F \in \mathrm{Coh}(M \times S)$ is obtained from a universal object.

2.4. **Some facts about curves.** We recall some of the consequences of the Serre duality and index theorem for curves. Let $C$ be a smooth projective connected curve over $\mathbb{C}$. Genus of a curve $C$ can be defined in several equivalent ways.

(1) Topological genus: Define $g$ such that $\dim_{\mathbb{Z}} H^1(C,\mathbb{Z}) = 2g$.
(2) Geometric genus: Define $g$ such that $\dim_{\mathbb{C}} H^0(C,K_C) = g$
(3) Arithmetic genus: Define $g$ such that $\dim_{\mathbb{C}} H^1(C,\mathcal{O}_C) = g$.

Equivalence between geometric genus and arithmetic genus is due to a Serre duality

$$H^0(C,K_C) \simeq H^1(C,\mathcal{O}_C)^\vee.$$

Equivalence with topological genus follows from a Hodge decomposition

$$H^1(C,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^1(C,\mathcal{O}_C) \oplus H^0(C,K_C).$$

On the other hand, by Riemann-Roch formula we have

$$1-g = \chi(C,\mathcal{O}_C) = \int_C \mathrm{td}(T_C) = \int_C \frac{c_1(T_C)}{2} = \frac{-\deg K_C}{2},$$

hence $\deg(K_C) = 2g-2$. It is convenient to remember a formula for a todd class

$$\mathrm{td}(T_C) = 1 + (1-g)[\mathrm{pt}]$$

where $[\mathrm{pt}] \in H^2(C,\mathbb{Z})$ is a Poincare dual to the point class. If we have a rank $r$ degree $d$ vector bundle $V$, Riemann-Roch formula says

$$\chi(C,V) = \int_C \mathrm{ch}(V)\cdot \mathrm{td}(C) = \int_C (r + d[\mathrm{pt}])(1+(1-g)[\mathrm{pt}]) = d + r(1-g).$$

(Exercise: compute $\chi(V,V)$ using Hirzebruch-Riemann-Roch)

## 3. Classification problem for sheaves

In this section, we motivate the classification problem for sheaves and explain what difficulties exist. Let $X$ be a smooth projective connected variety over $\mathbb{C}$. For such $X$, one can try to "classify" all the coherent sheaves on $X$. In other words, we would like to understand a category $\mathrm{Coh}(X)$ of coherent sheaves. When $X$ is just a point, coherent sheaves are simply finite dimensional vector spaces. So $\mathrm{Coh}(\mathrm{pt})$ is classified by a rank of a vector space. However for general $X$ it is not clear what it means by classifying $\mathrm{Coh}(X)$. We wish to make this question more accurate in this section.

3.1. **Skyscraper sheaves.** We start by restricting to a much smaller subset

$$\{\text{skyscraper sheaves } k(x)\} \subset \mathrm{Coh}(X).$$

A *set* of skyscraper sheaves is just a set of closed points $|X|$. At least in this example, it seems only natural to give a *geometric* structure to a set $|X|$ of skyscraper sheaves, namely a scheme structure of $X$. The question is how to justify such a intuition to upgrade a set to a geometric structure using the language of algebraic geometry.

As a first step toward to this, we need to make a definition below.

**Definition 4.** A *family of coherent sheaves* on $X$ is a $S$-flat coherent sheaf $\mathcal{F}$ on $S \times X$. For such a family, we say that $S$ is a parameter space for coherent sheaves $\{F_s\}_{s\in|S|}$.

**Remark 5.** Note that even if we modify a family $\mathcal{F}$ by $p^*L \otimes \mathcal{F}$ for some line bundle $L \in \mathsf{Pic}(S)$, this does not change the elements we are parametrizing because $L_s \otimes F_s \simeq F_s$ where $L_s$ is a rank 1 vector space.

Continuing from the previous example of skyscraper sheaves, we may ask if there is a flat family that parametrizes all the skyscraper sheaves. Consider a coherent sheaf $\mathcal{O}_\Delta$ on $X \times X$. First of all, $\mathcal{O}_\Delta$ is clearly flat over the first factor. Also we have for each $x \in |X|$ that

$$\mathcal{O}_\Delta\Big|_{\{x\}\times X} \simeq k(x).$$

Therefore, $\mathcal{O}_\Delta$ is an $X$-family parametrizing all the skyscraper sheaves. This suggests that a scheme structure $X$ is a right choice of geometric structure for $|X| = \{\text{skyscraper sheaves}\}$. To make this even more precise, we prove that every family of skyscraper sheaves are essentially coming from this family.

**Proposition 6.** *Let $\mathcal{F}$ be a $S$-flat family of skyscraper sheaves. Then there exists a morphism $f : S \to X$ and a line bundle $L \in \mathsf{Pic}(S)$ such that $(f \times \mathrm{id}_X)^*\mathcal{O}_\Delta \simeq p^*L \otimes \mathcal{F}$.*

Note that this property uniquely (up to ambiguity as in Remark 5) characterizes a family $\mathcal{O}_\Delta$ over $X \times X$. Therefore this justifies our intuition behind upgrading $|X|$ to $X$. In the later section, we will learn that this proposition can be rephrased as "$X$ is a *fine moduli space* of skyscraper sheaves with a *universal sheaf* $\mathcal{O}_\Delta$".

*Proof.* Suppose that we are given such a family $\mathcal{F}$. By definition, for every $s \in |S|$ we know that $F_s$ is a skyscraper sheaf at some point of $X$. This implies that

$$H^0(X, F_s) \simeq \mathbb{C} \quad \text{and} \quad H^i(X, F_s) = 0,\ i > 0.$$

By cohomology and base change, we know that

$$p_*\mathcal{F} : \text{line bundle}\ , \quad R^i p_*\mathcal{F} = 0,\ i > 0.$$

Since we are working up to an ambiguity of a line bundle from $S$, we may assume that $\mathcal{O}_S \xrightarrow{\sim} p_*\mathcal{F}$ by replacing $\mathcal{F}$ with its twist by a dual of $p_*\mathcal{F}$. By adjunction, we obtain a morphism $\phi : \mathcal{O}_{S\times X} \to \mathcal{F}$. We claim that this is in fact a surjection. We may check surjectivity on each fibers $\{s\} \times X$, namely the surjectivity of $\phi_s : \mathcal{O}_X \to F_s$. Since $F_s$ is a skyscraper sheaf, this amounts to show that $\phi_s \neq 0$. This follows from the fact that $\phi$ was an adjunction morphism of the isomorphism. Therefore, we may write $\mathcal{F} = \mathcal{O}_Z$ for a $S$-flat subscheme $Z \hookrightarrow S \times X$. Since $Z$ is flat over $S$ such that its restriction to each fiber $\{s\} \times X$ is a reduced point, we conclude that a composition $Z \hookrightarrow S \times X \to S$ is an isomorphism. By inverting this, we obtain a morphism

$$\mathrm{id}_S \times f : S \hookrightarrow S \times X$$

and $\mathcal{F} = (\mathrm{id}_S \times f)_* \mathcal{O}_S$. It is this $f$ that satisfies the desired identity with $L = \mathcal{O}_S$. □

3.2. **Topological classification problem.** We begin by the following proposition which in part justifies the notion of flatness of the family. Recall that $X$ was a smooth projective variety.

**Proposition 7.** *Let $\mathcal{F}$ be a $S$-flat family of coherent sheaves. Then for any $s_1, s_2 \in |S|$ in a same connected component, we have* $\mathrm{ch}_k(F_{s_1}) = \mathrm{ch}_k(F_{s_2})$.

*Proof.* Since $\mathcal{F}$ is a $S$-flat coherent sheaf over a smooth projective morphism $p : S \times X \to S$, we have a finite length resolution by vector bundles

$$0 \to V_d \to V_{d-1} \to \cdots \to V_0 \to \mathcal{F} \to 0.$$

At each $s \in |S|$, this induces

$$0 \to V_d\Big|_s \to V_{d-1}\Big|_s \to \cdots \to V_0\Big|_s \to F_s \to 0$$

hence

$$\mathrm{ch}(F_s) = \sum_{i=0}^{d} (-1)^i \mathrm{ch}(V_i\Big|_s) \in H^*(X, \mathbb{Q}).$$

Therefore it suffices to show that $\mathrm{ch}(V_s)$ for a vector bundle $V$ stays constant when $s$ varies in the same connected component. This is obvious because chern classes only depend on a topological structure of $V_s$ rather than algebraic structure. □

For example, we have $\mathrm{ch}(k(x)) = (0, \cdots, 0, [\mathrm{pt}]) \in H^*(X, \mathbb{Q})$ that does not depend on $x \in |X|$.

Topological classification problem asks what topological data is realized as a chern character of a coherent sheaf. That is, we want to understand what is the image

$$\Gamma_X := \mathrm{image}\Big(\mathrm{ch} : \mathrm{Coh}(X) \to H^*(X, \mathbb{Q})\Big).$$

We know that the image is contained in $(p,p)$-parts of the cohomology due to the fact that there is an algebraic version of the chern character. We also know that if $v \in H^*(X, \mathbb{Q})$ is in the image then first non zero degree must be a positive combination of integral cycles. However, finding the exact image is very difficult. In fact, when we replace $\mathrm{Coh}(X)$ by $K^0(X, \mathbb{Q})$ (which makes the problem only easier!), then this is equivalent to a Hodge conjecture.

However, when $X$ is a smooth projective curve, the image can easily be found. We know that

$$(r, d\,[\mathrm{pt}]) \in \Gamma_X = \mathrm{image}\Big(\mathrm{ch} : \mathrm{Coh}(X) \to H^*(X, \mathbb{Q})\Big)$$

if and only if

(1) $r \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}$ or
(2) $r = 0$, $d \in \mathbb{Z}_{\geq 0}$.

3.3. **Algebraic classification problem.** Recall that our goal was to classify $\mathrm{Coh}(X)$. Thanks to Proposition 7, this problem is divided into [9]

$$\mathrm{Coh}(X) = \coprod_{v \in \Gamma_X} \mathrm{Coh}(X)_v.$$

where $\mathrm{Coh}(X)_v$ consists of coherent sheaves with a fixed topological type $v \in \Gamma_X$. When $v = (0, 0, \cdots, 0, [\mathrm{pt}])$, we have seen that $\mathrm{Coh}(X)_v = \{\text{skyscraper sheaves}\}$ behaves very nicely in the sense that there is a universal family. We can ask a same question for general $v$.

**Question 8.** *Is there a "nice" scheme $S$ and a $S$-flat family of sheaves $\mathcal{F}$ that parametrizes all sheaves in* $\mathrm{Coh}(X)_v$ *such that every other family is pulled back from this up to an ambiguity of a line bundle from the parameter space?*

To stay within the realm of algebraic geometry, we require for the moduli space $S$ as above to be at least *of finite type* and *separated*. Unlike the case of skyscraper sheaves, we show that non of these properties are satisfied in general.

We can ask the question of finite type and separatedness for $\mathrm{Coh}(X)_v$ without necessarily having a moduli space in the sense of Question 8.

**Definition 9.** We say that $\mathrm{Coh}(X)_v$ is *of finite type* if there is a scheme $S$ of finite type and a $S$-flat family $\mathcal{F}$ of sheaves in $\mathrm{Coh}(X)_v$ such that $\{F_s\}_{s \in |S|} = \mathrm{Coh}(X)_v$.

**Definition 10.** We say that $\mathrm{Coh}(X)_v$ is *separated* if for any $\Delta = \mathrm{Spec}(\mathrm{DVR})$ and a $\Delta^*$-family $\mathcal{F}$ of sheaves in $\mathrm{Coh}(X)_v$, there exists at most one extension of $\mathcal{F}$ to a family of sheaves in $\mathrm{Coh}(X)_v$ over $\Delta$. [10]

In what follows, we show that neither of these properties are satisfied in general.

[9] In a technical term, we say that a moduli stack $\mathrm{Coh}(X)$ is a disjoint union of open substack $\mathrm{Coh}(X)_v$'s.
[10] Here we denote by $\Delta^*$ a generic point of a DVR $\Delta$. Notation is motivated from a punctured disk.

3.3.1. *Not of finite type.* Let $X = \mathbb{P}^1$ and $v = (2, 0)$. In other words, we are considering coherent sheaves on $\mathbb{P}^1$ of rank 2 and degree 0. For a contradiction, suppose that there is a scheme $S$ of finite type and a $S$-flat family $\mathcal{F}$ such that

$$\{F_s\}_{s\in|S|} = \mathrm{Coh}(X)_v.$$

By finiteness of $S$ and Serre vanishing property, there is an integer $m >> 0$ such that for every $s \in |S|$ we have that $H^1(\mathbb{P}^1, F_s(m)) = 0$. On the other hand, consider vector bundles

$$V_n := \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n), \quad n \geq 0.$$

They are of rank 2 and degree 0, hence an element of $\mathrm{Coh}(X)_v$. However, we have that

$$\begin{aligned} H^1(\mathbb{P}^1, V_n(m)) &\supseteq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-n+m)) \\ &\simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-m-2))^\vee \\ &\neq 0 \end{aligned}$$

for $n \geq m+2$. This gives a contradiction. It is worth mentioning that it was the "unbalancedness" of $V_n$ that caused unexpectedly large number of sections. We will see that how the notion of (semi)stability in the next lecture will remedy this situation by throwing away all these unbalanced vector bundles $V_n$.

3.3.2. *Not separated.* We show that $\mathrm{Coh}(X)_v$ in the previous example is also not separated. Consider the following exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\binom{y}{-x}} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \xrightarrow{(x\ y)} \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

This corresponds to a nonzero extension class

$$\xi \in \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq H^1(K_{\mathbb{P}^1}) \simeq \mathbb{C}.$$

Recall that the Ext group $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{C}$ parametrizes all possible extensions. This observation in fact glues giving a universal $\mathbb{A}^1$-flat family of extensions

$$0 \to \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{F} \to \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \to 0$$

over $\mathbb{A}^1 \times \mathbb{P}^1$. This is universal in the sense that for each $\epsilon \in \mathbb{A}^1$, this family of extension restricts at $\epsilon$ to a extension that represents the extension class $\epsilon \in \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq \mathbb{C}$. Therefore $\mathbb{A}^1$-flat family $\mathcal{F}$ satisfies that

$$\mathcal{F}_\epsilon \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} & , \quad \text{if } \epsilon \neq 0, \\ \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1), & \quad \text{if } \epsilon = 0. \end{cases}$$

This implies that $\mathrm{Coh}(X)_v$ is not separated because we have a constant family over $\mathbb{A}^1 \backslash \{0\}$ parametrizing $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ which can be extended across entire $\mathbb{A}^1$ in two different ways, namely the constant family and the above $\mathcal{F}$. Note again that non-separatedness was also happening because a balanced object $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ degenerated into the unbalanced one $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. This will also be remedied (in part) by the notion of semistability.

## 4. (Semi)stability of sheaves

Recall from the previous section that $\mathrm{Coh}(X)_v$ is neither of finite type nor separated in general. This was due to existence of "unbalanced" sheaves in $\mathrm{Coh}(X)_v$. In this section, we introduce a notion of (semi)stability in the case of smooth projective curve $X$. This produces a subcategory

$$\mathrm{Coh}(X)_v^{ss} \subset \mathrm{Coh}(X)_v$$

that has properties below that are to be explained in detail soon.

(1) Harder-Narasimhan filtration: $\mathrm{Coh}(X)_v$ is built uniquely from semistable pieces $\mathrm{Coh}(X)_{v_i}^{ss}$.
(2) Openness of semistability: $\mathrm{Coh}(X)_v^{ss}$ is an open subfunctor of $\mathrm{Coh}(X)_v$.
(3) Of finite type: $\mathrm{Coh}(X)_v^{ss}$ is of finite type.
(4) Almost separated: $\mathrm{Coh}(X)_v^{ss}$ is separated up to $S$-equivalence.

By the first property, $\mathrm{Coh}(X)_v^{ss}$ can be considered as building blocks for entire $\mathrm{Coh}(X)$. This justifies that we restrict our attention to semistable sheaves. The second property says that this restriction is a natural condition in algebraic geometry (as it is open). The third and fourth properties remedy exactly the problems that $\mathrm{Coh}(X)_v$ had.

### 4.1. **(Semi)stability.** Let $X$ be a smooth projective connected curve.

**Definition 11.** Let $F$ be a nonzero coherent sheaf on $X$.

(1) We define *slope* of $F$ as $\mu(F) := \frac{\deg(F)}{\mathrm{rk}(F)} \in (-\infty, \infty]$.
(2) We say that $F$ is *(semi)stable* if for every $0 \neq G \subsetneq F$ we have $\mu(G)\,(\leq)\,\mu(F)$.

**Remark 12.** We observe a few basic facts from the definition of (semi)stability.

(1) All zero dimensional sheaves are semistable.
(2) All semistable sheaves of positive rank are necessarily torion-free, hence locally free.
(3) Semistability is preserved by tensoring with a line bundle.
(4) If rank and degree are coprime, then semistability and stability coincides.

**Example 13.** All line bundles and skyscraper sheaves are stable sheaves. We will show in Proposition 17 that self direct summation preserves the semistability. Therefore $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ is semistable though not stable. However, the unbalanced examples $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ for $n > 0$ are not semistable.

There is an equivalent definition of slope and (semi)stability that is more geometric. Define a *stability function*

$$Z : K^0(X) \to \mathbb{C}, \quad F \mapsto (-\deg(F), \mathrm{rk}(F)).$$

This is clearly a group homomorphism. By positivity of the first non-zero chern character, stability function restricts to

$$Z : \mathrm{Coh}(X)\backslash\{0\} \to \mathbb{H} := \left\{ r \cdot e^{i\pi\phi} \,\middle|\, r > 0,\ 0 < \phi \leq 1 \right\}.$$

Note that there is a well-defined *phase function*

$$\phi : \mathbb{H} \to (0, 1].$$

One can check that there is an order preserving bijection between slopes $\in (-\infty, \infty]$ and phases $\in (0, 1]$. Therefore we can equivalently define (semi)stability using the phase function: a nonzero coherent sheaf $F$ is (semi)stable if and only if for every $0 \neq G \subsetneq F$ we have $\phi(G)\,(\leq)\,\phi(F)$. We will see the advantage of having more geometric definition of semistability via phase in the next section.

Even though we defined (semi)stability only for curves, there is a more general story for any projective scheme. However, what was equivalent for curves are no longer the same in higher dimensions. Slope (semi)stability generalizes to what's called Gieseker $H$-semistability. This is rather classical and much is known about it. Phase definition generalizes to what's called Bridgeland stability condition on any derived category of smooth projective variety. Brideland stability condition is rather subtle and even its existence is conjectural already for threefolds.

We prove basic yet important properties about (semi)stability.

**Proposition 14.** *Let $F_1$ and $F_2$ are semistable sheaves with $\mu(F_1) > \mu(F_2)$. Then* $\mathrm{Hom}(F_1, F_2) = 0$.

*Proof.* Suppose that there is a nonzero morphism $f : F_1 \to F_2$. This factors as

$$F_1 \twoheadrightarrow G \hookrightarrow F_2, \quad G \neq 0$$

which implies $\mu(F_1) \leq \mu(G) \leq \mu(F_2)$ hence the contradiction. □

**Proposition 15.** *Let $F_1$ and $F_2$ are stable sheaves with $\mu(F_1) = \mu(F_2)$. Then every $0 \neq f \in$* $\mathrm{Hom}(F_1, F_2)$ *is an isomorphism.*

*Proof.* As before, we get a factorization

$$F_1 \twoheadrightarrow G \hookrightarrow F_2, \quad G \neq 0$$

which implies $\mu(F_1) \leq \mu(G) \leq \mu(F_2)$. By assumption, we have $\mu(F_1) = \mu(G) = \mu(F_2)$. By stability, $F_1 \twoheadrightarrow G$ is necessarily an isormorphism and so is $G \hookrightarrow F_2$. □

**Proposition 16.** *Let $F$ be a stable sheaf. Then* $\mathrm{Hom}(F, F) \simeq \mathbb{C}$*, i.e., $F$ is simple.*

*Proof.* By the previous proposition, every nonzero element in $\mathrm{Hom}(F, F)$ is an isomorphism. In other words, the endomorphism ring $\mathrm{Hom}(F, F)$ is a finite dimensional division algebra over $\mathbb{C}$. Since $\mathbb{C}$ is an algebraically closed, it is same as $\mathbb{C}$. □

**Proposition 17.** *For any fixed slope $\mu \in (-\infty, \infty]$,* $\mathrm{Coh}(X)^{ss}_{\mu}$ *forms an abelian subcategory.*

*Proof.* We need to show two things:

(1) If $A, B \in \mathrm{Coh}(X)^{ss}_{\mu}$ and $f \in \mathrm{Hom}(A, B)$, then $\ker(f)$, $\mathrm{coker}(f) \in \mathrm{Coh}(X)^{ss}_{\mu}$.
(2) If we are given $A, C \in \mathrm{Coh}(X)^{ss}_{\mu}$ and $0 \to A \to B \to C \to 0$, then $B \in \mathrm{Coh}(X)^{ss}_{\mu}$.

For the first statement, consider the induced sequences

$$0 \to \ker(f) \to A \to \mathrm{image}(f) \to 0, \quad 0 \to \mathrm{image}(f) \to B \to \mathrm{coker}(f) \to 0.$$

By semistability of $A$ and $B$ we have $\mu(A) \leq \mu(\mathrm{image}(f)) \leq \mu(B)$ hence $\mu(\mathrm{image}(f)) = \mu$. This further implies that $\mu(\ker(f)) = \mu(\mathrm{coker}(f)) = \mu$. If $\ker(f)$ has a destabilizing subsheaf, the same sheaf destabilizes $A$. So $\ker(f)$ is semistable and so is $\mathrm{coker}(f)$, $\mathrm{image}(f)$ in the same way.

For the second statement, let $B' \hookrightarrow B$ be a subsheaf. Consider the induced map $f : B' \to C$ and denote the $\ker(f) := A'$ and $\mathrm{image}(f) := C'$. Then we have a exact sequence

$$0 \to A' \to B' \to C' \to 0$$

where $A' \hookrightarrow A$ and $C \hookrightarrow C$. By semistability, $\mu(A'), \mu(C') \leq \mu$ hence $\mu(B') \leq \mu$. This proves the semistability of $B$. □

That being said, we have a collection of abelian subcategories

$$\{\mathrm{Coh}(X)^{ss}_{\mu}\}_{\mu \in (-\infty, \infty]}$$

with property that

$$\mathrm{Hom}\Big(\mathrm{Coh}(X)^{ss}_{\mu_1}, \mathrm{Coh}(X)^{ss}_{\mu_2}\Big) = 0 \quad \text{if } \mu_1 > \mu_2.$$

These subcategories uniquely "generates" the entire category $\mathrm{Coh}(X)$ in the sense of Harder-Narasimhan filtration as we will see in the next section. We say that

$$\{\mathrm{Coh}(X)^{ss}_{\mu}\}_{\mu \in (-\infty, \infty]}$$

gives a *slicing* of $\mathrm{Coh}(X)$.

4.2. **Harder-Narasimhan filtration.**

**Theorem 18.** *For any nonzero coherent sheaf $F$, there exists a unique increasing filtration*

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_\ell = F$$

*such that*

1. $F_i/F_{i-1}$ *is semistable for all* $i = 1, \ldots, \ell$,
2. $\mu(F_1/F_0) > \mu(F_2/F_1) > \cdots > \mu(F_\ell/F_{\ell-1})$.

*Proof.* We first prove the uniqueness of the filtration. Suppose that we are given two filtrations $F_\bullet$ and $F'_\bullet$. We may assume that $\mu(F_1) \leq \mu(F'_1)$. Let $i$ be minimal with $F'_1 \subseteq F_i$. Then we have

$$F'_1 \to F_i \to F_i/F_{i-1}$$

which is non-trivial morphism by minimality of $i$. Since it is a non-trivial morphism between semistable sheaves, Proposition 14 implies that $\mu(F'_1) \leq \mu(F_i/F_{i-1})$. Therefore we have

$$\mu(F_i/F_{i-1}) \leq \mu(F_1) \leq \mu(F'_1) \leq \mu(F_i/F_{i-1}).$$

hence equality holds everywhere. This implies that $i = 1$, i.e. $F'_1 \subseteq F_1$. Repeating the argument with role of $F_\bullet$ and $F'_\bullet$ exchanged, we also obtain that $F_1 \subseteq F'_1$ showing that $F_1 = F'_1$. By induction, we prove the desired uniqueness of the HN filtration.

Now we prove the existence of HN filtration. We use the definition semistability via phase function. Since rank 0 sheaf is auotomatically semistable, we assume that $\mathrm{rk}(F) > 0$. Define *Harder-Narasimhan polygon* as below:

$$\mathrm{HN}(F) := \text{convex hull of } Z(G) \text{ for all } 0 \subseteq G \subseteq F.$$

We note several facts to understand this polygon geometrically.

1. We have $0 \in \mathrm{HN}(F)$ and $Z(F) := (-d, r) \in \mathrm{HN}(F)$.
2. For any $0 \subseteq G \subseteq F$, we have $\mathrm{rk}(G) \leq r$.
3. If $G$ is a rank $r$ subsheaf the $\deg(G) \leq d$. Conversely, for each $d' \leq d$ we can find subsheaf of rank $r$ and degree $d'$. Therefore, $\mathrm{HN}(F)$ contains the entire right hand side of the line segment joining 0 and $Z(F)$.
4. There is a constant $M \in \mathbb{Z}$ such that for every $0 \subseteq G \subseteq F$ we have $\deg(G) \leq M$.[11] Therefore, polygon $\mathrm{HN}(F)$ is left-bounded by $-M$.

The above facts restrict the geometry of convex polygon $\mathrm{HN}(F)$ quite dramatically. In particular, it is a polygon with only finitely many vertices.

[11]Hint: Find a semistable bundle surjecting onto $F$ using the projectivity of $X$. Then surjection of $F$ induces a surjection of this semistable bundle.

Vertices lying on the left hand side of the line segment joining 0 and $Z(F)$ are exactly those destabilizing $F$. Denote such vertices as $0 = z_0, z_1, \dots, z_\ell = Z(F)$ in the increasing order of phases. By finiteness of the polygon, we have $0 \subseteq F_i \subseteq F$ such that $z_i = Z(F_i)$. Clearly $F_0 = 0$ and $F_\ell = F$. We show that these $F_i$'s form HN filtration by checking the properties below:

(1) $F_{i-1} \subset F_i$ for all $i$.
(2) $\mu(F_1/F_0) > \cdots > \mu(F_\ell/F_{\ell-1})$.
(3) $F_i/F_{i-1}$ is semistable for all $i$.

For the first statement, consider

$$0 \to F_{i-1} \cap F_i \to F_{i-1} \oplus F_i \to F_{i-1} + F_i \to 0$$

where we denote the first and the third term by $A$ and $B$, respectively. Since $A$ and $B$ are subsheaf of $F$, we have $Z(A), Z(B) \in \mathrm{HN}(F)$. On the other hand, the exact sequence gives

$$\frac{Z(A) + Z(B)}{2} = \frac{z_{i-1} + z_i}{2}.$$

Since $z_{i-1}$ and $z_i$ were vertices of the convex polygon, this implies that $Z(A), Z(B) \in \overline{z_{i-1} z_i}$. Since $A \subseteq F_{i-1}$, rank of $A$ is less than or equal to that of $F_{i-1}$. This forces $Z(A) = z_{i-1} = Z(F_{i-1})$ hence $A = F_{i-1}$. Therefore, $F_{i-1} \subset F_i$.

The second statement also follows easily from the convex geometry because

$$Z(F_i/F_{i-1}) = z_i - z_{i-1}$$

and phases of $z_i - z_{i-1}$ are in increasing order.

If the third statement is false, then we have

$$F_{i-1} \subsetneq A \subsetneq F_i$$

such that $\mu(A/F_{i-1}) > \mu(F_i/F_{i-1})$. This means that $Z(A)$ stays outside of the convex polygon $\mathrm{HN}(F)$ which contradicts the fact that $A \subset F$. $\square$

4.3. **Openness of semistability in family.** We prove a geometric feature of a subcategory

$$\mathrm{Coh}(X)_v^{ss} \subset \mathrm{Coh}(X)_v.$$

**Theorem 19.** *Let $S$ be a finite type scheme and $\mathcal{F}$ be a $S$-flat family of sheaves on $X$ of chern character $v$. Then*

$$S^{ss} := \left\{ s \in |S| \,\middle|\, F_s \in \mathrm{Coh}(X)_v^{ss} \right\}$$

*is a Zariski open subset of $S$. The same is true if we replace semistability with stability.*

To prove this theorem, we use (without a proof) a classical construction of Grothendieck's Quot scheme. Rather than giving a most general form, we only state the version we need. See the survey paper [N] for more details on the Quot schemes.

**Theorem 20.** *Let $\mathcal{F}$ be a $S$-flat family of sheaves on $X$ of chern character $v$. Fix another chern character $v'$. There is a projective morphism*

$$\mathsf{Quot}_{S\times X/S}(\mathcal{F}, v') \to S$$

*and a universal quotient*

$$q^*\mathcal{F} \twoheadrightarrow \mathcal{G}$$

*over $\mathsf{Quot} \times_S (S \times X)$. Over each fiber $s \in |S|$, the Quot scheme parametrizes all the quotients*

$$F_s \twoheadrightarrow G, \quad \mathrm{ch}(G) = v'$$

*up to the usual equivalence.*

*Proof.* We prove openness of the semistability condition. Since $X$ is projective and $S$ is of finite type, there is a large $n$ such that for every $s \in |S|$ we have a surjection

$$\mathcal{O}_X(-n)^{\oplus N} \twoheadrightarrow F_s.$$

To check whether $F_s$ is semistable or not, we need to see if there is a quotient $F_s \twoheadrightarrow G$ with $\mu(v) > \mu(G)$. Since $\mathcal{O}_X(-n)^{\oplus N}$ is semistable, we always have

$$\mu(\mathcal{O}_X(-n)^{\oplus N}) \le \mu(G)$$

for any quotient of $F_s$. In other words, there are only finitely many $v' = \mathrm{ch}(G)$ that can potentially destabilize $F_s$ for some $s \in |S|$. Denote $I$ for a collection of such finitely many $v'$. Then we have

$$S^{ss} = S \setminus \bigcup_{v' \in I} \mathrm{image}\Big(\mathsf{Quot}_{S\times X/S}(\mathcal{F}, v') \to S\Big).$$

By projectivity of the relative Quot scheme and finiteness of $I$, we conclude that $S^{ss} \subseteq S$ is Zariski open. □

4.4. **Boundedness of $\mathrm{Coh}(X)^{ss}_v$.** We show that $\mathrm{Coh}(X)^{ss}_v$ is of "finite type" unlike $\mathrm{Coh}(X)_v$.

**Theorem 21.** *There is a scheme $S$ of finite type and a $S$-flat family of sheaves $\mathcal{F}$ such that*

$$\mathrm{Coh}(X)^{ss}_v = \{F_s\}_{s\in|S|}.$$

*Proof.* We first prove that there exist $n >> 0$ such that $F(n)$ is globally generated for every $F \in \mathrm{Coh}(X)^{ss}_v$. If $v$ was a class of zero dimensional sheaf, then it holds with even $n = 0$.[12] We

[12]Hint: proof by induction on the length of zero dimensional sheaves.

may assume that $v = (r, d)$ with $r > 0$. Recall that $F \in \mathrm{Coh}(X)^{ss}_v$ is necessarily a vector bundle. Consider any $\mathrm{pt} \in |X|$ and a ideal sequence

$$0 \to \mathcal{O}_X(-\mathrm{pt}) \to \mathcal{O}_X \to k(\mathrm{pt}) \to 0.$$

Since $F(n)$ is a vector bundle, we obtain another exact sequence

$$0 \to F(n) \otimes \mathcal{O}_X(-\mathrm{pt}) \to F(n) \to F(n)\Big|_{\mathrm{pt}} \to 0.$$

To show that $F(n)$ is globally generated, it suffices to show that

$$H^1(X, F(n) \otimes \mathcal{O}_X(-\mathrm{pt})) = 0$$

for every $\mathrm{pt} \in |X|$. By Serre duality, this is equivalent to the vanishing of

$$\mathrm{Hom}(F(n), K_X \otimes \mathcal{O}_X(\mathrm{pt})) = 0.$$

Since both $F(n)$ and $K_X \otimes \mathcal{O}_X(\mathrm{pt})$ are semistable, the Hom space vanishes as long as the slope of the first coordinate is bigger than the second coordinate. Note that this is a numerical criterion that does not depend on a choice of $F$ nor pt. Therefore, we have a uniform constant $n >> 0$ that does the job. Note that we may also assume that $H^1(X, F(n)) = 0$ for all $F \in \mathrm{Coh}(X)^{ss}_v$.

By the above argument, we have $n$ such that every $F \in \mathrm{Coh}(X)^{ss}_v$ is obtained as a quotient

$$H^0(X, F(n)) \otimes \mathcal{O}_X(-n) \twoheadrightarrow F \to 0.$$

Since the dimension

$$\dim H^0(X, F(n)) = \chi(X, F(n))$$

is independent on $F \in \mathrm{Coh}(X)^{ss}_v$, we may identify it with a fixed vector space, say $V$. Therefore we have

$$V \otimes \mathcal{O}_X(-n) \twoheadrightarrow F \to 0.$$

By projectivity (in particular, of finite type) of the Quot scheme

$$\mathsf{Quot}_X(V \otimes \mathcal{O}_X(-n), v)$$

and the flat universal quotient over $\mathsf{Quot} \times X$, the theorem follows after restricting to the open loci of Quot scheme whose quotient sheaf is semistable. □

4.5. **Separatedness of $\mathrm{Coh}(X)^{ss}_v$ up to $S$-equivalence.** Recall that semistable sheaves are building blocks for any coherent sheaves via Harder-Narasimhan filtration. Theorem below says that stable sheaves are building blocks for any semistable sheaves via *Jordan-Holder filtration.*

**Theorem 22.** *For any nonzero semistable sheaf $F$ of slope $\mu$, there exists a increasing filtration*

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_\ell = F$$

*such that graded piece $F_i/F_{i-1}$ is stable with slope $\mu$ for all $i = 1, \dots, \ell$. Even though such a filtration is not necessarily unique, the filtration length $\ell$ and the direct sum of the graded pieces*

$$gr(F) := \bigoplus_{i=1}^{\ell} F_i/F_{i-1}$$

*are well-defined.*

*Proof.* If every graded pieces of a filtration has the same slope $\mu$, they are necessarily semistable. If there is any strictly semistable graded piece then one can refine the filtration. Any such maximal refinement gives a Jordan-Holder filtration.

Now we prove that the length of the filtration and the direct sum of the graded pieces are well-defined. If $\mu = \infty$ or equivalently $F$ were zero dimensional sheaf, then $gr(F) = \oplus k(p_i)^{\oplus n_i}$ where $p_i$'s are distinct support of $F$ with length $n_i$. So we are done. Suppose now $F$ is a semistable bundle of rank $r > 0$ and degree $d$ and we are given two Jordan-Holder filtrations $F_\bullet$ and $F'_\bullet$ of length $\ell$ and $\ell'$. Let $i$ be minimal with $F'_1 \subseteq F_i$. Then we have

$$F'_1 \to F_i \to F_i/F_{i-1}$$

which is non-trivial morphism by minimality of $i$. Since $F'_1$ and $F_i/F_{i-1}$ are both stable sheaves of the same slope, the composition is an isomorphism. This isomorphism provides a splitting for the short exact sequence

$$0 \to F_{i-1} \to F_i \to F_i/F_{i-1} \simeq F'_1 \to 0.$$

Using this splitting we can construct two filtrations for $F/F'_1$. By induction on the rank $r = \mathrm{rk}(F)$, we may assume the well-definiteness statement for $F/F'_1$. Comparing the two Jordan-Holder filtrations whose exact definition we leave for the reader, we obtain that

$$\ell' - 1 = \ell - 1, \qquad \bigoplus_{1 \le i \le \ell',\ i \ne 2} F'_j/F'_{j-1} \simeq \bigoplus_{1 \le j \le \ell,\ j \ne i} F_j/F_{j-1}.$$

Since $F'_1 \simeq F_i/F_{i-1}$, we are done. □

**Definition 23.** We say that two semistable sheaves $F_1$ and $F_2$ are *$S$-equivalent* if $gr(F_1) \simeq gr(F_2)$.

In every $S$-equivalence class, we have a canonical representative, called polystable sheaf.

**Definition 24.** We say $F$ is *polystable* if $F \simeq \oplus F_i$ where $F_i$'s are stable sheaves with $\mu(F_i) = \mu(F)$.

We finish this section by a theorem on what it means by $\mathrm{Coh}(X)_v^{ss}$ is separated up to $S$-equivalence.

**Theorem 25.** *Let $\mathcal{F}_1, \mathcal{F}_2$ be two families of sheaves in $\mathrm{Coh}(X)_v^{ss}$ parametrized by $\Delta = \mathrm{Spec}(\mathrm{DVR})$. Suppose that $\mathcal{F}_1, \mathcal{F}_2$ agree over the generic point $\Delta^*$. Then at the closed point $0 \in \Delta$, two sheaves $\mathcal{F}_1\big|_0$ and $\mathcal{F}_2\big|_0$ are $S$-equivalent.*

*Proof.* We delay the proof until the construction of the moduli space. □

## 5. Geometric invariant theory

Geometric invariant theory is a powerful tool that is often used to construct a moduli space. We give a short introduction to geometric invariant theory. See [MFK] for details and proofs.

5.1. **What is a quotient?** Let $G$ be an algebraic group acting on a scheme $X$ of finite type/$\mathbb{C}$. We wish to construct a "quotient $X/G$" in a category of scheme of finite type. There are various notions for the quotient; *categorical quotient*, *good quotient* and *geometric quotient* written in an increasing order of being geometric.

We begin with a categorical quotient.

**Definition 26.** We say $f : X \to Y$ is a *categorical quotient* if it satisfies the properties below:

(1) $f : X \to Y$ is $G$-invariant.
(2) For any $G$-invariant morphism $f' : X \to Y'$, there is a unique morphism $g : Y \to Y'$ such that $f' = g \circ f$.

**Remark 27.** In other words, categorical quotient is a universal object among all $G$-invariant morphisms. Therefore categorical quotient is unique if it exists. In an abstract language, categorical quotient is a scheme that corepresents a functor

$$\underline{X/G} : \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set}, \quad S \mapsto X(S)/G(S)$$

where $X(S) := \mathrm{Mor}(S, X)$. This is because a functor $\underline{X/G} \to Y$ induces a $G$-invariant morphism $X \to Y$ and vice versa. We will see what it means by a scheme that corepresents a functor in the next lecture. By means of the universal property, we can check that properties below are preserved for a categorical quotient:

$$\text{connected}, \ \text{reduced}, \ \text{irreducible}, \ \text{normal}.$$

Categorical notion for the quotient is very neat in its formulation, but it is far from our experience in topology as we see below.

**Example 28.** Consider $\mathbb{C}^*$-action on $\mathbb{A}^n$ defined as $t \cdot (x_1, \ldots, x_n) := (tx_1, \ldots, tx_n)$. We claim that a morphism $f : \mathbb{A}^n \to \mathrm{Spec}(\mathbb{C})$ is a categorical quotient. This is clearly $\mathbb{C}^*$-invariant. Suppose that $f' : \mathbb{A}^n \to Y'$ is $\mathbb{C}^*$-invariant. Then $f'$ is constant along the closure of the orbits by the invariance and continuity. On the other hand, closure of the orbits are either rays or the origin. Since they always intersect, $f'$ must be a constant morhpism. In other words, $f'$ uniquely factors through $f$.

Even though this example was very far from topological intuition, it is natural from the algebraic point of view in the following sense: $f : \mathbb{A}^n \to \mathrm{Spec}(\mathbb{C})$ corresponds to the invariant subring

$$\mathbb{C} = \mathbb{C}[x_1, \ldots, x_n]^{\mathbb{C}^*} \hookrightarrow \mathbb{C}[x_1, \ldots, x_n].$$

This algebraic property (properties 1,2) together with some basic topological properties (properties 3,4,5) gives the next definition for the quotient.

**Definition 29.** We say $f : X \to Y$ is a *good quotient* if it satisfies the properties below:

(1) $f$ is an affine $G$-invariant morphism.
(2) The natural map $f^* : \mathcal{O}_Y \to \left(f_*\mathcal{O}_X\right)^G$ is an isomorphism.
(3) $f$ is a quotient map in Zariski topology.[13]
(4) For every $G$-invariant closed subset $V \subset X$, the image $f(V) \subset Y$ is also closed.
(5) If $V_1$ and $V_2$ are disjoint closed $G$-invariant subsets of $X$, then $f(V_1) \cap f(V_2) = \emptyset$.

**Remark 30.** If a good quotient $X \to Y$ exists, then closed points in $Y$ are in bijection to closed orbits in $X$. This follows essentially from the property (5) of the good quotient together with certain orbit stratification structure.

**Proposition 31.** *If $f : X \to Y$ is a good quotient, then it is also a categorical quotient, hence unique.*

*Proof.* Let $f' : X \to Y'$ be any $G$-invariant morphism. Let $\{V_i\}_{i\in I}$ be an affine open covering of $Y'$ and $W_i := X \setminus f'^{-1}(V_i)$ be a $G$-invariant closed subset of $X$. Define an open subset $U_i := Y \setminus f(W_i) \subseteq Y$.

Claim: $\{U_i\}_{i\in I}$ form an open covering of $Y$.

Suppose not, i.e., $\cap_{i\in I} f(W_i) \neq \emptyset$. In particular, we have $f(W_{i_1}) \cap \cdots \cap f(W_{i_N}) \neq \emptyset$ hence $W_{i_1} \cap \cdots \cap W_{i_N} \neq \emptyset$. On the other hand, we know that $\cap_{i\in I} W_i = \emptyset$. This is a contradiction since $X$ is a Noetherian topological space.

By construction, we have that $f^{-1}(U_i) \subseteq f'^{-1}(V_i)$. We want to construct a morphism $U_i \to V_i$. Since $V_i$ is affine, it suffices to define the following:

$$\begin{tikzcd}
\Gamma(V_i) \arrow[r, dotted, "g_i"] \arrow[d, "f'^*"'] & \Gamma(U_i) \arrow[d, equal, "f^*"] \\
\Gamma(f'^{-1}(V_i))^G \arrow[r, "\mathrm{res}"'] & \Gamma(f^{-1}(U_i))^G.
\end{tikzcd}$$

[13] A continuous map $f : X \to Y$ between topological spaces is called a *quotient map* if it is surjective and $U \subseteq Y$ is open iff $f^{-1}(U) \subseteq X$ is open.

We are left with checking that this morphism glues to a morphism $g : Y \to Y'$ with a desired property. We leave this as an excercise. $\square$

Main theorem of geometric invariant theory (on affine case) constructs a good quotient from the invariant subring.

**Theorem 32.** *Let $G$ be a reductive group acting on an affine scheme $X = \mathrm{Spec}(A)$. Then $Y = \mathrm{Spec}(A^G)$ is a scheme of finite type and a good quotient exists as a natural map*

$$f : \mathrm{Spec}(A) \to \mathrm{Spec}(A^G).$$

**Remark 33.** For this lecture, we do not need a precise definition of reductive group over $\mathbb{C}$. It suffices to know that basic examples such as $(\mathbb{C}^*)^n$, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{PGL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$ are reductive groups. An additive group $\mathbb{A}^1$ is an example of a non-reductive group.

**Example 34.** Consider $\mathbb{C}^*$-action on $\mathbb{A}^2$ defined as $t \cdot (x, y) := (tx, t^{-1}y)$. By the theorem above, good quotient (hence categorical quotient) is obtained from

$$\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}[xy] \hookrightarrow \mathbb{C}[x, y].$$

Geometrically, we have

$$f : \mathbb{A}^2 \to \mathbb{A}^1, \quad (x, y) \mapsto xy.$$

While all the fibers $f^{-1}(t)$ with $t \neq 0$ consists of exactly one orbit (which is closed), the central fiber $f^{-1}(0)$ consists three orbits

$$\{(0, 0)\}, \quad \mathbb{C}^* \times \{0\}, \quad \{0\} \times \mathbb{C}^*.$$

They lie over the same point because closure of these orbits intersect each other and out of those we have exactly one closed orbit, namely $\{(0, 0)\}$.

We define a most geometric version of the quotient.

**Definition 35.** We say $f : X \to Y$ is a *geometric quotient* if it is a good quotient such that $f^{-1}(y)$ is an orbit of $G$ for all $y \in |Y|$.

**Example 36.** Construction of a projective space $\mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ is a geometric quotient. It is actually something even better, called a principle $\mathbb{C}^*$-bundle.

5.2. **GIT for projective scheme.** Let $X$ be a projective scheme. Let $G$ be a reductive group

$$m : G \times G \to G$$

acting on $X$ via

$$\sigma : G \times X \to X.$$

Recall that a good quotient of the affine case was obtained by taking Spec of the invariant subring. To apply this approach in the projective case, we need extra data.

**Definition 37.** A *G-linearized ample line bundle* on $X$ is an ample line bundle $L$ together with an isomorphism $\phi : \sigma^* L \xrightarrow{\sim} p_2^* L$ on $G \times X$ satisfying the cocycle condition

$$p_{23}^* \phi \circ (\mathrm{id}_G \times \sigma)^* \phi = (m \times \mathrm{id}_X)^* \phi.$$

**Remark 38.** Roughly speaking, the isomorphism $\phi$ is a collection of isomorphisms

$$\phi_{g,x} : L_{g \cdot x} \xrightarrow{\sim} L_x.$$

Then cocycle condition says that the diagram below commutes:

$$\begin{tikzcd} L_{(gh)\cdot x} \arrow[rr, "\phi_{gh,x}"] \arrow[dr, "\phi_{g,h\cdot x}"'] & & L_x \\ & L_{h\cdot x} \arrow[ur, "\phi_{h,x}"'] & \end{tikzcd}$$

Note that a $G$-linearization of a line bundle $L$ induces a $G$-linearization on any tensor products $L^{\otimes n}$. Therefore the invariant subspaces $H^0(X, L^{\otimes n})^G$ is well-defined. By using the analogy of the affine case, we may define a quotient as

$$X = \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(X, L^{\otimes n})\right) \dashrightarrow \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(X, L^{\otimes n})^G\right)$$

Unlike the affine case, this morphism is only defined on some open subset of $X$. For each $s \in H^0(X, L^{\otimes n})^G$ with $n \geq 1$, define $U_s := \{x \in X \,|\, s(x) \neq 0\}$. Since $L$ is $G$-linearized ample line bundle, $U_s \subseteq X$ is an affine $G$-invariant open subset. Therefore we have a good quotient $f_s : U_s \to Y_s$. Define the *GIT-semistable locus* with respect to a $G$-linearized ample line bundle $L$ as $X^{ss}(L)$ as the union of such open subsets $U_s$. By the categorical property of the good quotient, we obtain a good quotient from the semistable locus

$$f^{ss} : X^{ss}(L) \longrightarrow \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(X, L^{\otimes n})^G\right) =: X^{ss} /\!/_L \, G.$$

We also define a locus where we actually obtains a geometric quotient. Define the *GIT-stable locus* with respect to a $G$-linearized ample line bundle $L$ as

$$X^s(L) := \left\{x \in X^{ss}(L) \,\middle|\, x \text{ has a finite stabilizer and a closed orbit in } X^{ss}(L)\right\}$$

Over this locus, we have a geometric quotient

$$f^s : X^s(L) \longrightarrow X^s /\!/_L \, G.$$

We summarize the main result for the projective case below.

**Theorem 39.** *Let $G$ be a reductive group acting on a projective scheme $X$. Let $L$ be a $G$-linearized ample line bundle. There is a projective scheme $Y$ with a good quotient*

$$\pi : X^{ss}(L) \to Y.$$

*Moreover, there is an open subset $Y^s \subseteq Y$ such that $\pi$ restricts to a geometric quotient*

$$\pi^s : X^s(L) \to Y^s.$$

**Remark 40.** Recall that two points $x_1, x_2 \in X^{ss}(L)$ are mapped to the same point under the good quotient map $\pi : X^{ss}(L) \to Y$ if and only if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq 0$ where we take the closure in $X^{ss}(L)$, not in $X$. In such a situation, we say that $x_1$ and $x_2$ are *$S$-equivalent.* Therefore, we have a bijection between

$$\big|X^{ss}(L)\big|/S\text{-equivalence} \quad \longleftrightarrow \quad \big|Y\big|.$$

We say that a point $x \in X^{ss}(L)$ is *polystable* if $\overline{G \cdot x} = G \cdot x$ in $X^{ss}(L)$. Since each closure of the orbit contains a unique closed orbit, we also have a bijection between polystable points $\big|X^{ps}(L)\big|$ and $|Y|$.

5.3. **Hilbet-Mumford criterion.** In specific examples, it is important to identify the GIT-(semi)stable open locus

$$X^s(L) \subseteq X^{ss}(L) \subseteq X.$$

Checking GIT-(semi)stability directly through the definition is difficult. We explain here Hilbert-Mumford numerical criterion that often corresponds to a very geometric interpretation.

Fix a point $x \in |X|$. We explain criterion for whether $x$ is GIT-(semi)stable with respect to $L$. Let $\lambda : \mathbb{G}_m \to G$ be a non-trivial one parameter subgroup of $G$. This induces a morphism

$$\mathbb{G}_m \to X, \quad t \mapsto \lambda(t) \cdot x.$$

By projectivity of $X$, there is a unique extension of this morphism to $\mathbb{G}_m \subseteq \mathbb{A}^1$. Denote the image of $0 \in \mathbb{A}^1$ as a limit

$$\overline{x} := \lim_{t \to 0} \lambda(t) \cdot x \in |X|.$$

It is easy to check that the limit point belongs to a fixed point $\overline{x} \in X^{\mathbb{G}_m}$ via $\lambda$. Therefore $\mathbb{G}_m$ acts on the fiber $L_{\overline{x}}$, hence defining the weight

$$\mu^L(x, \lambda) \in \mathbb{Z}.$$

**Theorem 41.** *A point $x \in |X|$ is GIT-(semi)stable with respect to $L$ if and only if for all non-trivial one parameter subgroup $\lambda$ of $G$, we have*

$$\mu^L(x, \lambda) \ (\geq) \ 0.$$

**Example 42.** Let $X = \mathbb{P}^2$ and $G = \mathbb{C}^*$. Let $\mathbb{C}^*$ act on $\mathbb{P}^2$ by $t \cdot [x_0 : x_1 : x_2] := [t^{-1}x_0 : x_1 : x_2]$. Note that an ample line bundle $L = \mathcal{O}_{\mathbb{P}^2}(1)$ admits infinitely many different $\mathbb{C}^*$-linearization. We fix one so that the universal subsheaf

$$L^\vee \hookrightarrow \mathcal{O}_{\mathbb{P}^2} \otimes (\mathbb{C}\mathbf{t}^{-1} \oplus \mathbb{C} \oplus \mathbb{C})$$

is a $\mathbb{C}^*$-equivariant morphism. Here $\mathbf{t}$ denotes the weight one representation of $\mathbb{C}^*$. For this choice, we claim that

$$X^{ss}(L) = \mathbb{P}^2 \backslash \{[1:0,0]\}, \quad X^s(L) = \emptyset.$$

Let $\lambda : \mathbb{C}^* \to \mathbb{C}^*$ be any non-trivial one parameter subgroup. It is always of the form

$$\lambda : \mathbb{C}^* \to \mathbb{C}^*, \quad t \mapsto t^n$$

for some $n \neq 0$. Let $p = [x_0 : x_1 : x_2] \in \mathbb{P}^2$ be a point. To analyze the weight at the limit point, we divide the problem into three cases:

$$1)\ x_0 = 0, \quad 2)\ x_0 \neq 0 \text{ and } (x_1, x_2) = (0,0), \quad 3)\ x_0 \neq 0 \text{ and } (x_1, x_2) \neq (0,0).$$

In the first case, the limit point is

$$\overline{p} := \lim_{t \to 0} [t^{-n} \cdot 0 : x_1, x_2] = [0 : x_1 : x_2].$$

By restricting the universal subsheaf to a point $\overline{p}$, we obtain

$$L^\vee\Big|_{\overline{p}} \hookrightarrow \mathbb{C}\mathbf{t}^{-n} \oplus \mathbb{C} \oplus \mathbb{C}$$

where the map is given by a coordinate itself $\overline{p} = [0 : x_1 : x_2]$. Therefore $L_{\overline{p}}$ has a weight 0 via $\lambda$ hence semistable but not stable.

In the second case, the limit point is

$$\overline{p} := \lim_{t \to 0} [t^{-n}x_0 : 0 : 0] = [1 : 0 : 0].$$

By restricting the universal subsheaf to a point $\overline{p}$, we obtain

$$L^\vee\Big|_{\overline{p}} \hookrightarrow \mathbb{C}\mathbf{t}^{-n} \oplus \mathbb{C} \oplus \mathbb{C}$$

where the morphism embeds into the first factor. Therefore $L_{\overline{p}}$ has a weight $n$. Since $n$ can be any nonzero integer, such a point is not semistable.

In the third case, the limit point is

$$\overline{p} := \lim_{t \to 0} [t^{-n}x_0 : x_1 : x_2] = \begin{cases} [1 : 0 : 0] \quad , & \text{if } n > 0, \\ [0 : x_1 : x_2], & \text{if } n < 0. \end{cases}$$

Using the similar argument as before, the weight of $L_{\overline{p}}$ is $n$ if $n > 0$ and 0 if $n < 0$. Therefore such a point is semistable but not stable.

Therefore GIT quotient gives a projective quotient

$$X^{ss}(L) \simeq \mathbb{P}^2 \backslash \{[1:0:0\} \to X^{ss} /\!/_L G$$

where

$$X^{ss} /\!/_L G = \operatorname{Proj}\left( \mathbb{C}[x_0, x_1, x_2]^{\mathbb{C}^*} \right) = \operatorname{Proj}\left( \mathbb{C}[x_1, x_2] \right) \simeq \mathbb{P}^1_{[x_1:x_2]}.$$

Geometrically, this good quotient is a radial projection away from $[1:0:0]$ to the line at infinity.We recommend the reader to check how GIT quotient changes if one choose a different $\mathbb{C}^*$-linearization for the same group action. For example, one can consider a $\mathbb{C}^*$-linearlization for $L = \mathcal{O}_{\mathbb{P}^2}(1)$ such that

$$L^\vee \hookrightarrow \mathcal{O}_{\mathbb{P}^2} \otimes (\mathbb{C} \oplus \mathbb{C}\mathbf{t} \oplus \mathbb{C}\mathbf{t})$$

becomes a $\mathbb{C}^*$-equivariant morphims.

## 6. Construction of the moduli space

6.1. **Moduli functors.** To properly introduce a notion of moduli space in algebraic geometry, we need some category theory. Let $\mathsf{Sch}$ be a category schemes of finite type over $\mathbb{C}$. We consider a Zariski site for the category $\mathsf{Sch}$. Rather than considering geometric objects directly, we may instead deal with corresponding functors (also called as presheaves)

$$\mathcal{M} : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}.$$

We define a category $\mathsf{Psh}$, called a *category of presheaves* on $\mathsf{Sch}$, of all such functors whose morphisms are givey by natural transformations between functors. We say that a presheaf $\mathcal{M} \in \mathsf{Psh}$ is a *sheaf on a Zariski site* if for any Zariski open covering $\{S_i \to S\}$ the diagram

$$\mathcal{M}(S) \to \prod_i \mathcal{M}(S_i) \rightrightarrows \prod_{i,j} \mathcal{M}(S_{ij})$$

is an equalizer. Denote the full subcategory $i : \mathsf{Sh} \hookrightarrow \mathsf{Psh}$ of all sheaves. This embedding $i$ has a left adjoint functor $(-)^+ : \mathsf{Psh} \to \mathsf{Sh}$, called a *sheafification.* We say that a natural transformation $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a *local isomorphism* in Zariski site if the induced morphism $\phi^+ : \mathcal{M}_1^+ \to \mathcal{M}_2^+$ is an isomorphism. More concretely, $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a local isomorphism if and only if for every $S \in \mathsf{Sch}$ the following two conditions are satisfied:

(1) Given $x, y \in \mathcal{M}_1(S)$ such that $\phi(x) = \phi(y) \in \mathcal{M}_2(S)$, there is an open covering $S = \cup S_i$ such that $x\Big|_{S_i} = y\Big|_{S_i}$ for all $i$.
(2) Given $x \in \mathcal{M}_2(S)$, there is an open covering $S = \cup S_i$ and $x_i \in \mathcal{M}_1(S_i)$ such that $\phi(x_i) = b\Big|_{S_i}$ for all $i$.

We will see an example of locally isomorphic presheaves which are not globally isomorphic.

The most important notion in the category theory is Yoneda lemma. This says that there is a fully faithful embedding

$$\mathsf{Sch} \hookrightarrow \mathsf{Psh}, \quad X \mapsto \mathrm{Mor}(-, X)$$

whose essential image is called as representable functors. In this sense, we extended our original geometric category $\mathsf{Sch}$ into $\mathsf{Psh}$. It is clear that a representable functors are sheaves in Zariski site.

**Example 43.** There are numerous sources for the functors, i.e., presheaves.

(1) All the representable functors.

(2) Let $G$ be an algebraic group acting on a scheme $X \in \mathsf{Sch}$. This defines a quotient functor

$$\underline{X/G} : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}, \quad S \mapsto X(S)/G(S).$$

Here we used a standard notation $X(S) := \mathrm{Mor}(S, X)$ for a $S$-valued points in $X$.

(3) Fix a finite dimensional vector space $V$ of rank $n$ and an integer $0 < k < n$. Define a functor $\mathrm{Gr}(V, k) : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ such that

$$S \mapsto \{V \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{W} \,|\, \mathcal{W} : \text{locally free sheaf of rank } k\}/\sim .$$

As the notation suggests, this functor is represented by a Grassmannian $\mathrm{Gr}(V, k)$.

(4) Fix a smooth projective scheme $X$, a coherent sheaf $H \in \mathrm{Coh}(X)$ and a topological data $v \in H^*(X, \mathbb{Q})$. Define a functor $\mathsf{Quot}_X(H, v) : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ such that

$$S \mapsto \{q^*H \twoheadrightarrow \mathcal{F} \,|\, \mathcal{F} : S\text{-flat family of sheaves on } X \text{ with chern character } v\}/\sim .$$

This functor is represented by Grothendieck's Quot scheme which is known to be projective.

(5) Let $X$ be a smooth projective connected curve and $v = (r, d)$. Define a functor

$$\mathrm{Coh}(X)^{ss}_v : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}, \quad S \mapsto \{S\text{-flat family } \mathcal{F} \text{ of semistable sheaves on } X \text{ with chern character } v\}/\sim$$

where $\mathcal{F}_1 \sim \mathcal{F}_2$ if they are isomorphic. We call this a moduli functor for semistable sheaves on $X$ with chern character $v$. Similarly, one can define it for the stable sheaves.

(6) In the setting of the previous example, we can define a projective liner version for the moduli functor $\mathrm{pl.Coh}(X)^{ss}_v : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ such that

$$S \mapsto \{S\text{-flat family } \mathcal{F} \text{ of semistable sheaves on } X \text{ with chern character } v\}/\sim$$

where $\mathcal{F}_1 \sim \mathcal{F}_2$ if there is a line bundle $L \in \mathsf{Pic}(S)$ such that $\mathcal{F}_1 \simeq q^*L \otimes \mathcal{F}_2$. We all this a projective linear moduli functor.

To relate these functors to more geometic objects, namely schemes, we define the notion of (co)representability.

**Definition 44.** Let $\mathcal{M} : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ be a functor.

(1) A functor $\mathcal{M}$ is *represented* by a scheme $M$ with a natural transformation $\alpha : M \to \mathcal{M}$ if it is universal among such morphisms. In other words, we have a diagram

$$\begin{tikzcd} M \arrow[rr, "\alpha"] & & \mathcal{M} \\ & S \arrow[ul, dashed, "\exists!\gamma"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

for any scheme $S$ with a natural transformation $\beta : S \to \mathcal{M}$.

(2) A functor $\mathcal{M}$ is *corepresented* by a scheme $M$ with a natural transformation $\alpha : \mathcal{M} \to M$ if it is universal among such morphisms. In other words, we have a diagram

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\alpha"] \arrow[dr, "\beta"] & & M \arrow[dl, dashed, "\exists!\gamma"] \\ & S & \end{tikzcd}$$

for any scheme $S$ with a natural transformation $\beta : \mathcal{M} \to S$.

**Remark 45.** By universal property, (co)representing scheme is unique up to a unique isomorphism. If $\mathcal{M}$ is represented by $\alpha : M \to \mathcal{M}$, then $\alpha$ is necessarily an isomorphism, hence $M$ also corepresents $\mathcal{M}$. Representable functors are equipped with a unique universal object $\mathcal{U} \in \mathcal{M}(M)$ which is obtained as an imange of $\mathrm{id}_M$ via $\alpha$. This is not the case for corepresentability. It is a good exercise to check that a quotient functor $\underline{X/G}$ is corepresented by $Y$ if and only if there is a categorical quotient $X \to Y$. This provides examples of corepresentable functors which are not representable.

Now we can state what it means by a moduli space of semistable sheaves whose existence is what we are after.

**Theorem 46.** *Let $X$ be a smooth projective connected curve and $v = (r, d)$. A functor $\mathrm{Coh}(X)_v^{ss}$ is corepresented by a projective variety $M_X(r, d)$ whose points correspond to $S$-equivalence classes of semistable sheaves of class $v$. Furthermore, there is an open subset $M_X^s(r, d) \subseteq M_X(r, d)$ that corepresents a functor $\mathrm{Coh}(X)_v^s$.*

**Remark 47.** Recall that we have a projective linearization map between two functors

$$\pi : \mathrm{Coh}(X)_v^{ss} \to \mathrm{pl.Coh}(X)_v^{ss}.$$

One can check that this functor is an local isomorphism. Therefore corepresentability of these two functors are equivalent because they are same after sheafification. However, two functors behave very differently with respect to representability. First of all, $\mathrm{Coh}(X)_v^{ss}$ is never representable because universal sheaf (if exists) is not unique up to an isomorphism. On the other hand, we will see that $\mathrm{pl.Coh}(X)^{ss}_v$ is representable if $gcd(r,d)=1$, meaning that there is a universal sheaf unique up to a Picard group of the moduli space.

6.2. **The case of zero dimensional sheaves.** In Theorem 46, we are mostly interested in the case where $v=(r,d)$ with $r\geq 1$. For pedagogical point of view, however, it is worth starting with an easier case where $v=(0,d)$ with $d\geq 1$.

Let $F$ be any zero dimensional sheaf of length $d$, which is automatically semistable. Then $F$ is globally generated by $d$-dimensional sections $H^0(X,F)$. This puts $F$ into a quotient of the form

$$0\to K\to V\otimes\mathcal{O}_X\to F\to 0,\quad \dim(V)=d$$

such that a global section induces an isomorphism

$$H^0(X,V\otimes\mathcal{O}_X)\xrightarrow{\sim}H^0(X,F).$$

Motivated from this observation, consider a Quot scheme

$$\mathsf{Quot}=\mathsf{Quot}_X(V\otimes\mathcal{O}_X,(0,d))$$

and an open subset $R\subseteq\mathsf{Quot}$ parametrizing those quotients $[V\otimes\mathcal{O}_X\twoheadrightarrow F]$ inducing an isomorphism on the global sections. Note that there is a natural action on the Quot scheme

$$\mathrm{GL}(V)\curvearrowright\mathsf{Quot},\quad \rho\cdot[V\otimes\mathcal{O}_X\xrightarrow{f}F]\mapsto[V\otimes\mathcal{O}_X\xrightarrow{\rho^{-1}\otimes 1}V\otimes\mathcal{O}_X\xrightarrow{f}F].$$

[14] It is clear that $R$ is $\mathrm{GL}(V)$-invariant open subset. This construction gives a bijection

$$\underline{R/\mathrm{GL}(V)}(\mathbb{C})\leftrightarrow\mathrm{Coh}(X)^{ss}_{(0,d)}(\mathbb{C})$$

between $\mathbb{C}$-points of two functors. We claim that this can be upgraded into local isomorphism between functors.

**Lemma 48.** *We have a natural transformation*

$$\Phi:\underline{R/\mathrm{GL}(V)}\to\mathrm{Coh}(X)^{ss}_{(0,d)}$$

*that is a local isomorphism.*

*Proof.* Recall that we have a universal quotient

$$V\otimes\mathcal{O}_{R\times X}\twoheadrightarrow\mathcal{F}$$

over $R\times X$. For any scheme $S$, we can pull back the family over $R$ to define

$$R(S)\to\mathrm{Coh}(X)^{ss}_{(0,d)}(S).$$

[14]We use $\rho^{-1}$ to make this a left action rather than a right action.

To define a natural transformation, we are left to prove that this morphism factors through the quotient by $\mathrm{GL}(V)(S)$. But this is clear because $\mathrm{GL}(V)$-action does not change the universal sheaf $\mathcal{F}$. This construct a natural transformation

$$\Phi : \underline{R/\mathrm{GL}(V)} \to \mathrm{Coh}(X)^{ss}_{(0,d)}, \quad \overline{f} \mapsto (f \times \mathrm{id}_X)^* \mathcal{F}.$$

To show that $\Phi$ is a local isomorphism, we need to check two condition. For the first condition, pick any $\overline{f}, \overline{g} \in R(S)/\mathrm{GL}(V)(S)$ such that $(f \times \mathrm{id}_X)^* \mathcal{F} \simeq (g \times \mathrm{id}_X)^* \mathcal{F}$ where $f, g : S \to R$. Recall that $(f \times \mathrm{id}_X)^* \mathcal{F}$ is an $S$-flat family of zero dimensional sheaves. Since each member has vanishing higher cohomology and global generation, we have

$$p^* p_* \Big[ (f \times \mathrm{id}_X)^* \mathcal{F} \Big] \twoheadrightarrow (f \times \mathrm{id}_X)^* \mathcal{F}$$

where $p_* \Big[ (f \times \mathrm{id}_X)^* \mathcal{F} \Big]$ is a rank $d$ vector bundle on $S$. Pick an open covering $S = \cup S_i$ so that this vector bundle is trivialized over each $S_i$. In particular, we have

$$V \otimes \mathcal{O}_{S_i \times X} \simeq p^* p_* \big[ (f \times \mathrm{id}_X)^* \mathcal{F} \big] \Big|_{S_i \times X} \twoheadrightarrow (f_i \times \mathrm{id}_X)^* \mathcal{F}.$$

In the same way, we have

$$V \otimes \mathcal{O}_{S_i \times X} \simeq p^* p_* \big[ (g \times \mathrm{id}_X)^* \mathcal{F} \big] \Big|_{S_i \times X} \twoheadrightarrow (g_i \times \mathrm{id}_X)^* \mathcal{F}.$$

From the original isomorphism $(f \times \mathrm{id}_X)^* \mathcal{F} \simeq (g \times \mathrm{id}_X)^* \mathcal{F}$, we can show that these two quotients are the same, i.e., $\overline{f_i} = \overline{g_i} \in R(S)/\mathrm{GL}(V)(S_i)$. We leave for the reader to check the second part of the local isomorphism criterion.

$\square$

Since we have a local isomorphism between the moduli functor and the quotient functor, it suffices to prove the existence of the projective categorical quotient for the $\mathrm{GL}(V)$-action on $R$. This is where we use GIT construction. For that we need two things:

(1) We need to define a $\mathrm{GL}(V)$-linearized ample line bundle $\mathcal{L}$ on $\mathsf{Quot}$.
(2) We need to show that $\mathsf{Quot}^{ss}(\mathcal{L}) = R$ by comparing the GIT semistability (the left hand side) and the sheaf semistability (the right hand side) which is trivial in the case of zero dimensional sheaf.

We start with the first problem of defining $\mathrm{GL}(V)$-linearlization. To start the discussion, we fix an integer $\ell >> 0$. To each quotient of a sheaf $V \otimes \mathcal{O}_X$

$$0 \to K \to V \otimes \mathcal{O}_X \to F \to 0$$

in $\mathsf{Quot}$ we can associate the quotient of a vector space

$$0 \to H^0(X, K(\ell)) \to H^0(X, V \otimes \mathcal{O}_X(\ell)) \to H^0(X, F(\ell)) \to 0.$$

This remains a short exact sequence because we may choose $\ell >> 0$ so that all the higher cohomology groups below vanish

$$H^1(X, K(\ell)) = H^1(X, V \otimes \mathcal{O}_X(\ell)) = H^1(X, F(\ell)) = 0.$$

This gives a morphism

$$j : \mathsf{Quot} \to \mathrm{Gr}(W, d)$$

where $W$ is a vector space of dimension $h^0(X, V \otimes \mathcal{O}_X(\ell))$ and $d = h^0(X, F(\ell))$. It turns out that this morphism $j$ (depending on the choice of $\ell$) is a closed embedding. This fact is crucial in the proof of representability of Quot functors. Since $j$ is a $\mathrm{GL}(V)$-equivariant embedding and so is the Plucker embedding of $\mathrm{Gr}(W, d)$, we obtain a $\mathrm{GL}(V)$-linearization on the ample line bundle

$$\mathcal{L}_\ell := j^* \mathcal{O}_{\mathrm{Gr}(V,d)}(1).$$

The subscript $\ell$ emphasizes the dependence on $\ell$. If we trace back this construction, we can show that

$$\mathcal{L}_\ell = \det \Big( p_* \big( \mathcal{F} \otimes q^* \mathcal{O}_X(\ell) \big) \Big). \tag{1}$$

For the above choice of $\mathrm{GL}(V)$-linearized ample line bundle, we study GIT semistability. Note that $\mathbb{C}^* \subset \mathrm{GL}(V)$ acts trivially on $\mathsf{Quot}$, hence it suffices to consider $\mathrm{PGL}(V)$-action instead. Furthermore, we may use $\mathrm{SL}(V)$ because we have an exact sequence

$$0 \to \mu_d \to \mathrm{SL}(V) \to \mathrm{PGL}(V) \to 0.$$

For this $\mathrm{SL}(V)$-linearized ample line bundle $\mathcal{L}_\ell$, we show that

$$\mathsf{Quot}^{ss}(\mathcal{L}_\ell) = R.$$

We use Hilbert-Mumford numerical criterion. Let $\lambda : \mathbb{G}_m \to \mathrm{SL}(V)$ be a non-trivial one parameter subgroup. This is equivalent to have a non-trivial weight decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V_n \mathbf{t}^n \quad \text{such that} \quad \sum_{n \in \mathbb{Z}} n \cdot \dim(V_n) = 0.$$

Fix a point $p := [V \otimes \mathcal{O}_X \twoheadrightarrow F] \in \mathsf{Quot}$. Define the induced subobjects

$$V_{\le n} := \bigoplus_{i \le n} V_i, \quad F_{\le n} := \mathrm{image}(V_{\le n} \otimes \mathcal{O}_X \to F).$$

This defines a filtration of the original quotient and also the graded pieces

$$V_n \twoheadrightarrow F_n := F_{\le n}/F_{\le n-1}, \quad n \in \mathbb{Z}.$$

Let $\overline{p} \in \mathsf{Quot}$ be a point that represents the quotient

$$V \otimes \mathcal{O}_X \twoheadrightarrow \bigoplus_{n \in \mathbb{Z}} F_n =: \overline{F}.$$

We claim that this is the limit point in the sense that

$$\overline{p} := \lim_{t\to 0} \lambda(t)\cdot p \in \mathsf{Quot}.$$

To show this, we need to describe the corresponding $\mathbb{A}^1$-family of quotients such that over $t \neq 0$ it describes a quotient $\lambda(t)\cdot p$ and over $t = 0$ it specializes to $\overline{p}$. Using the increasing filtration structure, we define a quotient

$$\mathcal{V} := \bigoplus_{n\in\mathbb{Z}} V_{\leq n}\otimes \mathcal{O}_X \otimes \mathbb{C}\langle T^n\rangle \xrightarrow{\rho'} \mathcal{F} := \bigoplus_{n\in\mathbb{Z}} F_{\leq n}\otimes \mathbb{C}\langle T^n\rangle.$$

between coherent sheaves over $\mathbb{A}^1\times X$. The multiplication by $T$ is acting only on the second factor of the form $\mathbb{C}\langle T^n\rangle$. Coherence of the sheaves follows from the fact that negative enough summands vanish. On the other hand, we may define a morphism

$$V\otimes \mathbb{C}[T] \xrightarrow{\gamma} \mathcal{V} = \bigoplus_{n\in\mathbb{Z}} V_{\leq n}\otimes \mathbb{C}\langle T^n\rangle$$

between coherent sheaves over $\mathbb{A}^1$ defined as

$$v\otimes T^s = \sum_{n\in\mathbb{Z}} v_n\otimes T^s \mapsto \sum_{n\in\mathbb{Z}} v_n \otimes T^{n+s}.$$

This isomorphism describes exactly the action if $\lambda$ in the sense that over each $t\neq 0$ it restricts to $\lambda(t) : V \xrightarrow{\sim} V$. We leave the details to the reader to check that $\gamma$ is an isomorphism with this property. Combining these constructions, we define a quotient of the trivial vector bundle

$$\rho := \rho' \circ (\gamma\otimes \mathrm{id}_X) : V\otimes \mathcal{O}_X \otimes \mathbb{C}[T] \twoheadrightarrow \mathcal{F}.$$

It is straight forward to check that at $T = 0$, this quotient specializes to

$$V\otimes \mathcal{O}_X \to \bigoplus_{n\in\mathbb{Z}} F_n \mathbf{t}^n$$

that represents $\overline{p}$.

We now turn our attention to the computation of the weight

$$\mu^{\mathcal{L}_\ell}(p,\lambda) = \mathrm{weight}\Big(\mathbb{C}^* \curvearrowright \mathcal{L}_\ell\Big|_{\overline{p}}\Big)$$

Using the formula (1), we have

$$\begin{aligned}\mathcal{L}_\ell\Big|_{\overline{p}} &= \det\left(p_*\Big(\bigoplus_{n\in\mathbb{Z}} F_n(\ell)\,\mathbf{t}^n\Big)\right)\\ &= \bigotimes_{n\in\mathbb{Z}} \det\left(H^0(X, F_n(\ell))\mathbf{t}^n\right)\\ &= \mathbf{t}^{\sum_{n\in\mathbb{Z}} n\cdot \chi(X, F_n(\ell))},\end{aligned}$$

hence

$$\begin{aligned}
\mu^{\mathcal{L}_\ell}(p,\lambda) &= -\sum_{n\in\mathbb{Z}} n\cdot\chi(F_n(\ell)) \\
&= -\frac{1}{\dim(V)}\sum_{n\in\mathbb{Z}} n\Big(\dim(V)\chi(F_n(\ell)) - \dim(V_n)\chi(F(\ell))\Big) \\
&= \frac{1}{\dim(V)}\sum_{n\in\mathbb{Z}} \Big(\dim(V)\chi(F_{\le n}(\ell)) - \dim(V_{\le n})\chi(F(\ell))\Big) \\
&= \frac{1}{\dim(V)}\sum_{n\in\mathbb{Z}} \alpha(V_{\le n}).
\end{aligned}$$

Minus sign in the first equality comes from the fact that the left-action on the Quot scheme is defined as $\rho\cdot[V\otimes\mathcal{O}_X \xrightarrow{f} F]\mapsto [V\otimes\mathcal{O}_X \xrightarrow{\rho^{-1}\otimes 1} V\otimes\mathcal{O}_X \xrightarrow{f} F]$. In the last equality, we used a notation

$$\alpha(V') := \dim(V)\chi(F'(\ell)) - \dim(V')\chi(F(\ell))$$

for each subspace $0\subseteq V'\subseteq V$ and $F' := \operatorname{image}(V'\otimes\mathcal{O}_X\to F)$.

We claim that $p = [V\otimes\mathcal{O}_X \twoheadrightarrow F]$ is GIT (semi)stable if and only if we have $\alpha(V')(\ge)0$ for any $0\subsetneq V'\subsetneq V$. If direction is straightforward by the above computation and Hilbert-Mumford criterion. For the converse, we use the following construction. For each $0\subsetneq V'\subsetneq V$ we have a one parameter subgroup $\lambda$ that correspond to the decomposition

$$V = V'\mathbf{t}^{-\dim(V'')}\oplus V''\mathbf{t}^{\dim(V')}$$

where $V''$ is a choice of a complement of $V'$ in $V$. It is easy to check that for this $\lambda$ we have $\mu^{\mathcal{L}_\ell}(p,\lambda) = \alpha(V')$. Therefore GIT (semi)stability of a point $p$ implies that $\alpha(V')(\ge)0$.

In conclusion, we have proved that $p = [V\otimes\mathcal{O}_X\twoheadrightarrow F]\in\mathsf{Quot}$ is GIT semistable with respect to $\mathcal{L}_\ell$ if and only if we have

$$\dim(V)\chi(F'(\ell)) \ge \dim(V')\chi(F(\ell))$$

for every $0\subsetneq V'\subsetneq V$. Since $\dim(V) = \chi(F(\ell)) = d$, this is equivalent to $\chi(F'(\ell))\ge\dim(V')$. We claim that this condition is equivalent to $V = H^0(X, V\otimes\mathcal{O}_X)\to H^0(X,F)$ being an isomorphism, i.e., $p\in R\subseteq\mathsf{Quot}$. Suppose that the morphism $V\to H^0(X,F)$ is not an isomorphism. Since they have a same dimension, it must have a non-trivial kernel which we denote by $V'$. For such $V'$, we have $F' = 0$ hence $\chi(F'(\ell)) < \dim(V')$. For the converse, suppose that there exists a subspace $0\subsetneq V'\subsetneq V$ such that $\chi(F'(\ell)) < \dim(V')$. Consider a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \longrightarrow & V'\otimes\mathcal{O}_X & \longrightarrow & F' & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & V\otimes\mathcal{O}_X & \longrightarrow & F & \longrightarrow & 0.
\end{array}$$

Since $h^0(F'(\ell)) < \dim(V')$ we have $h^0(K') \neq 0$ hence $h^0(K) \neq 0$. Therefore $V \to H^0(X, F)$ cannot be an isomorphism. This proves the desired equality

$$\mathsf{Quot}^{ss}(\mathcal{L}_\ell) = R.$$

In the zero dimensional case, we can say something more about the projective moduli space $M_X(0, d)$. Consider a symmetric product $\mathrm{Sym}^d(X)$ parametrizing direct sum of the skyscraper sheaves $\oplus_{i=1}^d k(x_i)$, which are exactly polystable zero dimensional sheaves. This family will induce an element

$$\mathrm{Coh}(X)^{ss}_{(0,d)}(\mathrm{Sym}^d(X)).$$

Since $\mathrm{Coh}(X)^{ss}_{(0,d)}$ is corepresented by $M_X(0, d)$ this induces a morphism

$$\mathrm{Sym}^d(X) \to M_X(0, d).$$

On the other hand, we can define a natural transformation $\mathrm{Coh}(X)^{ss}_{(0,d)} \to \mathrm{Sym}^d(X)$. This requires the notion of fitting ideals which I will not explain in detail. In an example, this associate to $k(x) \oplus k(x)$ an ideal $\mathcal{O}_X(-2x) \hookrightarrow \mathcal{O}_X$ which is different from the annihilating ideal $\mathcal{O}_X(-x) \hookrightarrow \mathcal{O}_X$ that defines the scheme theoretic support. Fitting ideal behaves well with respect to family and pullbacks hence defining the above natural transformation. By the corepresentability, this factors through $M_X(0, d) \to \mathrm{Sym}^d(X)$. It is easy to show that these two morphisms are inverse to each other. So we conclude that $\mathrm{Coh}(X)^{ss}_{(0,d)}$ is correpresented by

$$M_X(0, d) \simeq \mathrm{Sym}^d(X).$$

6.3. **The case of positive ranks.** Now we come back to the case of most interest, namely $\mathrm{Coh}(X)^{ss}_{(r,d)}$ with $r > 0$. Most of the core ideas has been already appeared in the zero dimensional case and we will follow the similar path. Note that there is an equivalence between presheaves

$$\otimes L : \mathrm{Coh}(X)^{ss}_{(r,d)} \xrightarrow{\sim} \mathrm{Coh}(X)^{ss}_{(r,d+r)}$$

where $L$ is a degree 1 line bundle on $X$. This is because semistability is preserved by tensoring with a line bundle. Therefore we may choose $d$ as large as possible. We fix $d > d(g, r)$ a large enough $d$ that only depends on the rank $r$ and the genus $g$. Range of such $d$ will be specified as we proceed. By the proof of Theorem 21, if $F \in \mathrm{Coh}(X)^{ss}_{(r,d)}$ with $d > d(g, r)$ then $F$ is globally generated with $H^1(X, F) = 0$. One can be more precise about the range and show that it actually works for $d(g, r) = r(2g - 1)$. However we may have to choose $d$ even larger in the other steps of the proof. By Riemann-Roch formula, $h^0(X, F) = d + r(1 - g) =: N$. We fix a vector space $V$ of dimension $N$. From this argument, every $F \in \mathrm{Coh}(X)^{ss}_{(r,d)}$ is obtained from a certain quotient in the Quot scheme

$$\mathsf{Quot} := \mathsf{Quot}_X(V \otimes \mathcal{O}_X, (r, d)).$$

Let $R \subseteq \mathsf{Quot}$ be an open subset that corresponds to quotients $[V \otimes \mathcal{O}_X \twoheadrightarrow F]$ inducing an isomorphism on the global sections. Difference from the zero dimensional case is that we have to consider the open subsets

$$R^s \subseteq R^{ss} \subseteq R$$

corresponding to the quotients $[V \otimes \mathcal{O}_X \twoheadrightarrow F]$ with (semi)stable $F$. These are open subsets because (semi)stability is an open condition in a flat family. All these open subsets are clearly $\mathrm{GL}(V)$-invariant with respect to a natural $\mathrm{GL}(V)$-action on $\mathsf{Quot}$.

**Lemma 49.** *We have a natural transformation*

$$\underline{R^{ss}/\mathrm{GL}(V)} \to \mathrm{Coh}(X)^{ss}_{(r,d)}$$

*that is a local isomorphism. The same is true for the stable case.*

*Proof.* The proof is completely identical to the zero dimensional case. □

We prove corepresentability of the presheaf $\underline{R^{ss}/\mathrm{GL}(V)}$ using GIT. Again, we may replace the group by $\mathrm{SL}(V)$. Fix an integer $\ell > \ell(r, g, d)$ such that

$$H^1(X, K) = H^1(X, V \otimes \mathcal{O}_X) = H^1(X, F) = 0$$

for every quotient $[V \otimes \mathcal{O}_X \twoheadrightarrow F] \in \mathsf{Quot}$ with $K = \ker(V \otimes \mathcal{O}_X \to F)$. This then defines a closed embedding

$$j : \mathsf{Quot} \hookrightarrow \mathrm{Gr}\Big(H^0(X, V \otimes \mathcal{O}_X(\ell))\,,\, \chi(F(\ell))\Big)$$

defined as

$$\big[V \otimes \mathcal{O}_X \twoheadrightarrow F\big] \mapsto \big[H^0(X, V \otimes \mathcal{O}_X(\ell)) \twoheadrightarrow H^0(X, F(\ell))\big].$$

Pulling back the plucker line bundle through $j$, we define $\mathrm{SL}(V)$-linearized ample line bundle

$$\mathcal{L}_\ell := \det\Big(p_*\big(\mathcal{F} \otimes q^*\mathcal{O}_X(\ell)\big)\Big).$$

Let $\overline{R} \subseteq \mathsf{Quot}$ be a scheme theoretic closure of $R \subseteq \mathsf{Quot}$. It remains to prove the following theorem comparing GIT (semi)stability and sheaf (semi)stability.[15]

**Theorem 50.** *For $d > d(g, r)$ and $\ell > \ell(g, r, d)$, we have*

$$\overline{R}^{ss}(\mathcal{L}_\ell) = R^{ss}, \quad \overline{R}^{s}(\mathcal{L}_\ell) = R^s.$$

[15]Actually, we will prove that Theorem 50 is true even if we replace $\overline{R}$ with entire Quot scheme $\mathsf{Quot}$. We use $\overline{R}$ for more or less notational reason. However, working with $\overline{R}$ is strictly necessary for the construction of moduli of semistable bundles on higher dimensional varieties.

Once we prove this theorem, it follows that $\mathrm{Coh}(X)^{ss}_{(r,d)}$ is corepresented by a projective scheme $M_X(r,d)$ with an analogous statement for the stable case. Proof of this theorem will take several steps where we make the GIT (semi)stablility closer and closer to the sheaf (semi)stability. To be more precise, we prove that for $d > d(g,r)$, $\ell > \ell(g,r,d)$ and $p \in [V \otimes \mathcal{O}_X \to F] \in \overline{R}$ the following statements are all equivalent.

(I) A point $p$ is GIT (semi)stable.

(II) For every subspace $0 \subsetneq V' \subsetneq V$ we have

$$\dim(V) \cdot \chi(F'(\ell)) \;(\geq)\; \dim(V') \cdot \chi(F(\ell))$$

where $F' = \Phi(V')$.

(III) For every subspace $0 \subsetneq V' \subsetneq V$ we have

$$\dim(V) \cdot P(F') \;(\geq)\; \dim(V') \cdot P(F)$$

where $F' = \Phi(V')$ and $P(-)$ is the Hilbert polynomial.

(IV) For every subsheaf $0 \subset F' \subsetneq F$ with $V' := \Psi(F') \neq 0$, we have

$$\dim(V) \cdot P(F') \;(\geq)\; \dim(V') \cdot P(F).$$

(V) A point $p$ is sheaf (semi)stable, i.e., $F$ is semistable with $V \xrightarrow{\sim} H^0(X,F)$.

In the statements, $\Phi(V')$ is a subsheaf of $F$ generated by $V'$ and $\Psi(F')$ is a subspace of $V$ defined as a preimage of $H^0(X,F') \subseteq H^0(X,F)$ under the linear map $V \to H^0(X,F)$. We will study nice properties between $\Phi$ and $\Psi$ later.

Equivalence between (I) and (II) is due to Hilbert-Mumford criterion. Since the proof is exactly the same with the zero dimensional case, we leave the details to the reader.

Define a *Hilbert polynomial* of a sheaf $F$ with respect to an ample line bundle $\mathcal{O}_X(1)$ as

$$P(F) := \chi(X, F \otimes \mathcal{O}_X(\ell)).$$

This is indeed a polynomial in a variable $\ell$ of degree equal to $\dim(\mathrm{supp}(F))$ by Riemann-Roch formula.

Now we prove the equivalence between (II) and (III). Note that the family of subsheaves $F'$ generated by certain $0 \subsetneq V' \subsetneq V$ forms a bounded family once $(g,r,d)$ is fixed. On the other hand, inequality in (II) can be written as

$$\dim(V)\Big(r'\ell + d' + r'(1-g)\Big) \;(\geq)\; \dim(V')\Big(r\ell + d + r(1-g)\Big)$$

where $\mathrm{ch}(F') = (r', d')$. Since there are only finitely many collection of $(r', d')$ appearing in this inequality, we may choose $\ell > \ell(g,r,d)$ so that numerical inequality for a fixed $\ell$ is equivalent to the inequality between polynomials. This proves the equivalence between (II) and (III).

Note that there is no dependence on $\ell$ in the statements (III), (IV) and (V). Now we prove the equivalence between (III) and (IV) by changing the testing objects from subspaces $V'$ to subsheaves $F'$. Consider the monotonic maps between two ordered sets

$$\Phi : \{V' \mid 0 \subseteq V' \subseteq V\} \leftrightarrows \{F' \mid 0 \subseteq F' \subseteq F\} : \Psi$$

where $\Phi$ sends $V'$ to $\mathrm{image}(V' \otimes \mathcal{O}_X \to F)$ and $\Psi$ sends $F'$ to preimage of $H^0(X, F') \subseteq H^0(X, F)$ via $V \to H^0(X, F)$. One can check that $\Phi$ and $\Psi$ forms a "Galois connection" in the sense that for each $V'$ and $F'$ we have

$$\Phi(V') \subseteq F' \quad \Longleftrightarrow \quad V' \subseteq \Psi(F').$$

In particular, this implies that

$$V' \subseteq \Psi \circ \Phi(V'), \quad \Phi \circ \Psi(F') \subseteq F'.$$

Also note that $\Phi(V) = F$ and $\Psi(F) = V$. We may restrict the Galois connection to subsets

$$\Phi : A = \{V' \mid 0 \subsetneq V' \subsetneq V,\ \Phi(V') \neq F\} \leftrightarrows \{F' \mid 0 \subseteq F' \subsetneq F,\ \Psi(F') \neq 0\} = B : \Psi$$

On the other hand, in the statement (III) it suffices to check inequalities for $V' \in A$ because if $0 \subsetneq V' \subsetneq V$ with $\Phi(V') = F$ then we have

$$\dim(V) \cdot P(F) > \dim(V') \cdot P(F)$$

is automatically satisfied. Therefore, we need to prove that

$$V' \in A \implies \dim(V) \cdot P(\Phi(V')) \ (\geq) \ \dim(V') \cdot P(F)$$

if and only if

$$F' \in B \implies \dim(V) \cdot P(F') \ (\geq) \ \dim(\Psi(F')) \cdot P(F).$$

This follows from the general property of Galois connection and the fact that $\dim(-)$ and $P(-)$ are both monotonic functions. To prove the if direction, let $V' \in A$ and denote $F' := \Phi(V') \in B$. Then we have

$$\begin{aligned} \dim(V) \cdot P(F') &\ (\geq) \dim(\Psi(F')) \cdot P(F) \\ &= \dim(\Psi \circ \Phi(V')) \cdot P(F) \\ &\geq \dim(V') \cdot P(F). \end{aligned}$$

To prove the only if direction, let $F' \in B$ and denote $V' := \Psi(F') \in A$. Then we have

$$\begin{aligned} \dim(V) \cdot P(F') &\geq \dim(\Phi \circ \Psi(F')) \cdot P(F) \\ &= \dim(\Phi(V')) \cdot P(F) \\ &(\geq) \dim(V') \cdot P(F). \end{aligned}$$

Now we prove that (IV) implies (V). Suppose that $p = [V \otimes \mathcal{O}_X \twoheadrightarrow F]$ satisfies (IV). We need to show the followings.

(i) Higher cohomology $H^1(X, F) = 0$ vanishes.

(ii) Global section induces an isomorphism $V \xrightarrow{\sim} H^0(X, F)$.

(iii) $F$ is locally free.

(iv) $F$ is (semi)stable.

To prove (i), suppose that $H^1(X, F) \neq 0$. By Serre duality, we have a non-zero morphism $F \to K_X$ which we factor by $F \twoheadrightarrow F'' \subseteq K_X$. Denote $F' := \ker(F \twoheadrightarrow F'')$ and $V' := \Psi(F')$. Consider an exact sequence $0 \to F' \to F \to F'' \to 0$ and a diagram

$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & V'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(X, F') \arrow[r] & H^0(X, F) \arrow[r] & H^0(X, F'') & \end{tikzcd}$$

where $V'' := \mathrm{image}\big(V \to H^0(X, F') \to H^0(X, F'')\big)$. Then condition (IV) for $F'$

$$\dim(V) \cdot P(F') \ (\geq) \ \dim(V') \cdot P(F)$$

can be written as

$$\dim(V) \cdot P(F'') \ (\leq) \ \dim(V'') \cdot P(F).$$

By taking the leading coefficient, this implies that

$$r(1-g) + d = \dim(V) \leq r\dim(V'') \leq rg$$

where we used that $V'' \subseteq H^0(X, F'') \subseteq H^0(X, K_X)$. This implies that $d \leq r(2g-1)$ but we may choose $d > d(g, r)$ so that this does not happen. Therefore $H^1(X, F) = 0$.

To prove (ii), it suffices to prove that $V \to H^0(X, F)$ is injective because they have the same dimension by (i). Suppose that we have a non-trivial kernel $V' := \ker(V \to H^0(X, F))$. By definition, we have $F' := \Phi(V') = 0$. This contradicts condition (III) because

$$\dim(V) \cdot P(F') = 0 < \dim(V') \cdot P(F).$$

To prove (iii), suppose that there is a zero dimensional subsheaf $0 \subsetneq F' \subsetneq F$. Since $F'$ is zero dimensional we have $H^0(X, F') \neq 0$. This implies that $V' := \Psi(F') \neq 0$ because we know that $V \simeq H^0(X, F)$ from (ii). Therefore we have

$$\dim(V) \cdot P(F') < \dim(V') \cdot P(F)$$

for degree reason, contradicting condition (IV).

To prove (iv), suppose that $F$ is not (semi)stable. Then maximal slope subbundle $0 \subsetneq F' \subsetneq F$ satisfies $\mu(F') \, [\geq] \, \mu(F)$ where $[\geq]$ stands for $>$ for semistability and $\geq$ for stability. Since $F'$ is semistable bundle of slope at least $\mu(F)$ and rank at most $r-1$, we may choose $d > d(g, r)$ large enough so that $H^1(X,F')=0$ and $H^0(X,F')\neq 0$. Then $V' := \Psi(F')\neq 0$ since we know that $V \xrightarrow{\sim} H^0(X,F)$ from (ii). Therefore we may apply condition (IV) for this $F'$ to obtain

$$\chi(F)\cdot P(F')\ (\geq)\ \chi(F')\cdot P(F).$$

Note that we have two polynomials with the same constant term $\chi(F)\cdot\chi(F')$. Therefore, this is equivalent to the inequality between the leading coefficients

$$\chi(F)\cdot r' \geq \chi(F')\cdot r,$$

or equivalently $\mu(F)(\geq)\mu(F')$. This contradicts that $\mu(F')[\geq]\mu(F)$.

We are left to show that (V) implies (IV). For this, we need an upper bound on the number of global sections of locally free sheaves. We introduce a notation $[a]_+ := \max\{a,0\}$ for the next proposition.

**Proposition 51.** *Let $F$ be a locally free sheaf of* $\mathrm{ch}(F)=(r,d)$*. Then we have*

$$\frac{h^0(X,F)}{r}\leq\left(1-\frac{1}{r}\right)[\mu_{\max}(F)+1]_+ + \frac{1}{r}[\mu(F)+1]_+.$$

*Here $\mu(F)$ is the slope of $F$ and $\mu_{\max}(F)$ is the maximal slope of the semistable graded pieces in the Harder-Narasimhan filtration of $F$.*

*Proof.* We first consider the case of semistable locally free sheaf $F$, in which case the lemma says $h^0(X,F)/r\leq[\mu(F)+1]_+$. If $\mu(F)<0$ then the inequality follows from $h^0(X,F)=0$. For $\mu\geq 0$ we prove by induction on the degree of $F$. Consider a sequence

$$0\to F(-x)\to F\to F_x\to 0.$$

Using the long exact sequence, we have

$$h^0(X,F)\leq h^0(X,F(-x))+r.$$

Since $F(-x)$ is semistable locally free with strictly lower degree than $F$, we have

$$\frac{h^0(X,F(-x))}{r}\leq[\mu(F(-x))+1]_+ = [\mu(F)]_+$$

from the induction hypothesis. Combining these two inequalities, we have

$$\frac{h^0(X,F)}{r}\leq[\mu(F)]_+ + 1 = [\mu(F)+1]_+.$$

Now we consider the general case of locally free sheaf $F$. Let

$$0=F_0\subsetneq F_1\subsetneq\cdots\subsetneq F_\ell = F$$

be a Hardar-Narasimhan filtration of $F$. For each $i=1,\dots,\ell$, we have an exact sequence

$$0\to F_{i-1}\to F_i\to F_i/F_{i-1}\to 0.$$

Denote $G_i := F_i/F_{i-1}$ and $r_i = \mathrm{rk}(G_i)$ and $\mu_i := \mu(G_i)$. By the long exact sequence, we have

$$h^0(X, F_i) \leq h^0(X, F_{i-1}) + h^0(X, G_i).$$

Summing up all the inequalities for $i = 1, \ldots, \ell$, we obtain

$$\begin{aligned}
\frac{h^0(X,F)}{r} &\leq \sum_{i=1}^{\ell} \frac{r_i}{r} \cdot \frac{h^0(X,G_i)}{r_i} \\
&\leq \sum_{i=1}^{\ell} \frac{r_i}{r} \cdot [\mu_i + 1]_+ \\
&\leq \left( \sum_{i=1}^{\ell-1} \frac{r_i}{r} \cdot [\mu_{\max}(F) + 1]_+ \right) + \frac{r_\ell}{r} [\mu(F) + 1]_+ \\
&= \left( 1 - \frac{r_\ell}{r} \right) \cdot [\mu_{\max}(F) + 1]_+ + \frac{r_\ell}{r} [\mu(F) + 1]_+ \\
&\leq \left( 1 - \frac{1}{r} \right) \cdot [\mu_{\max}(F) + 1]_+ + \frac{1}{r} [\mu(F) + 1]_+
\end{aligned}$$

proving the statement. □

The lemma below clearly proves that (V) implies (IV).

**Lemma 52.** *Suppose that $d > d(g, r)$ and let $F$ be a (semi)stable sheaf with $\mathrm{ch}(F) = (r, d)$. Then for every $0 \subsetneq F' \subsetneq F$ we have*

$$h^0(F) \cdot P(F') \ (\geq) \ h^0(F') \cdot P(F).$$

*Proof.* Suppose that $F$ is (semi)stable sheaf with $\mathrm{ch}(F) = (r, d)$ where $d > d(g, r)$. Pick a constant $C$ in the range

$$rg < C < \min \left\{ \mu, \frac{\mu - (2g-2)}{r} \right\}.$$

We can always find such $C$ as long as $d$ is large enough compared to $r$ and $g$. Let $0 \subsetneq F' \subsetneq F$ be any subsheaf and denote $r' = \mathrm{rk}(F')$ and $\mu' = \mu(F')$. The lemma follows from the two statements below.

(1) If $\mu(F') < \mu - C$, then

$$h^0(F) \cdot r' > h^0(F') \cdot r$$

(2) If $\mu(F') \geq \mu - C$, then

$$h^0(F) \cdot r' \geq h^0(F') \cdot r$$

and if equality holds, then $\mu(F') = \mu(F)$.

We consider the first case, i.e., $\mu' < \mu - C$. Since $F$ is semistable, we also have $\mu_{\max}(F') \le \mu$. By the previous lemma, we have

$$\begin{aligned}
\frac{h^0(F')}{r'} &\le \left(1 - \frac{1}{r'}\right)[\mu_{\max}(F') + 1]_+ + \frac{1}{r'}[\mu' + 1]_+ \\
&\le \left(1 - \frac{1}{r}\right)[\mu + 1]_+ + \frac{1}{r}[\mu - C + 1]_+ \\
&= \mu + 1 - \frac{C}{r} \\
&< \mu + 1 - g \\
&= \frac{h^0(F)}{r}.
\end{aligned}$$

Now we consider the second case, i.e., $\mu' \ge \mu - C$. We claim that $H^1(X, F') = 0$. Once this is done, we need to show that

$$\chi(F) \cdot r' \ge \chi(F') \cdot r$$

with equality if and only if $\mu(F') = \mu(F)$ which follows from the semistability of $F$. By Serre duality, it suffices to show that $\mathrm{Hom}(F', K_X) = 0$. Suppose for the contradiction that there is a non-trivial morphism $F' \to K_X$. This factors through $F' \twoheadrightarrow L \hookrightarrow K_X$ for some line bundle $L$ of degree $n \le (2g-2)$. Let $K := \ker(F' \twoheadrightarrow L)$. Since $K$ is a subsheaf of semistable $F$, we see

$$\mu \ge \mu(K) = \frac{d' - n}{r' - 1} \ge \frac{d' - (2g-2)}{r' - 1}$$

hence

$$d' \le (2g-2) + r'\mu - \mu.$$

On the other hand, we are in the case with $\frac{d'}{r'} \ge \mu - C$, hence $d' \ge r'\mu - r'C$. Combining these two inequalities, we have

$$r'\mu - r'C \le (2g-2) + r'\mu - \mu$$

which implies

$$\mu - rC \le (2g-2).$$

This contradicts the choice of $C$. □

We have finally proven most of Theorem 46 except the following lemma.

**Lemma 53.** *Let $p = [V \otimes \mathcal{O}_X \twoheadrightarrow F]$ and $p' = [V \otimes \mathcal{O}_X \twoheadrightarrow F']$ be points in $\overline{R}^{ss}(\mathcal{L}_\ell) = R^{ss}$. Then $p$ and $p'$ are GIT S-equivalent if and only if $F$ and $F'$ are sheaf S-equivalent.*

*Proof.* Let $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_\ell = F$ be a Jordan-Holder filtration. Let $G_i = F_i/F_{i-1}$ for $i = 1, \dots, \ell$ which are stable sheaves of slope $\mu(F)$. By taking $d > d(g, r)$, we may assume that every $G_i$ is globally generated with $H^1(X, G_i) = 0$. By long exact sequence, this implies that each $F_i$ is also globally generated with $H^1(X, F_i) = 0$. Now let $V_{\le i} := \Psi(F_i)$. By the previous properties of $F_i$, we know that $\dim(V_{\leq i}) = h^0(X, F_i) = \chi(X, F_i)$ and $\Phi(V_{\leq i}) = F_i$. Choose $V_i \subseteq V_{\leq i}$ which splits the filtration of $V$. We can associate to this $V_\bullet$ an one parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{GL}(V)$[16] such that

$$\overline{p} := \lim_{t \to 0} \lambda(t) \cdot p \in \mathsf{Quot}$$

corresponds to a quotient given by a graded pieces

$$V \otimes \mathcal{O}_X = \bigoplus_{i \in \mathbb{Z}} V_i \otimes \mathcal{O}_X \twoheadrightarrow \bigoplus_{i \in \mathbb{Z}} (F_i / F_{i-1}) =: gr(F).$$

In fact, this limit point lives in the semistable locus $R^{ss}$. This construction proves that if $F$ and $F'$ are $S$-equivalent then $p$ and $p'$ are GIT $S$-equivalent because they share the same limit point (up to $\mathrm{GL}(V)$-action) under certain choice of one parameter subgroup.

For the other direction, it suffices to show that $p = [V \otimes \mathcal{O}_X \twoheadrightarrow F] \in R^{ss}$ is GIT polystable if $F$ is a polystable sheaf. Suppose that $\overline{p} := [V \otimes \mathcal{O}_X \twoheadrightarrow F']$ is in the closure of the orbit $\overline{G \cdot p}$ inside $R^{ss}$. It suffices to show that $F' \simeq F$ because a quotient $[V \otimes \mathcal{O}_X \to F]$ is unique up to $\mathrm{GL}(V)$-action for polystable $F$. By the assumption, there is a smooth pointed curve $(C, 0)$ parametrizing a flat family of quotients $V \otimes \mathcal{O}_{C \times X} \twoheadrightarrow \mathcal{F}$ where

$$\mathcal{F}\Big|_{(C \setminus 0) \times X} \simeq q^* F, \quad \mathcal{F}\Big|_{\{0\} \times X} \simeq F'.$$

Let $\{F_i\}_{i \in I}$ be a set of all stable sheaves with slope $\mu(F)$ and rank at most $r$. For each such $F_i$ we define $n_i := \hom(F_i, F)$ and $n_i' := \hom(F_i, F')$. By semicontinuity we have $n_i' \geq n_i$. Polystability of $F$ implies

$$F \simeq \bigoplus_{i \in I} F_i^{\oplus n_i}.$$

On the other hand, we can consider the evaluation map

$$\phi_i : F_i \otimes \mathrm{Hom}(F_i, F') \to F', \quad i \in I$$

which we claim to be injective. Suppose otherwise and let $K := \ker(\phi_i)$ be non-trivial. Since semistable sheaves of the same slope form an abelian category, we know that $K$ must be semistable with slope $\mu(F)$. Let $K_1$ be the first element in the Jordan-Holder filtration of $K$. Then $K_1$ must be isomorphic to $F_i$. This implies that there is a one dimensional subspace $W \subseteq \mathrm{Hom}(F_i, F')$ such that restriction of the evaluation map $\phi_i\Big|_W : F_i \otimes W \to F'$ is a zero map which is a contradiction. Therefore $\phi_i$ is an injection for each $i \in I$ hence we obtain

$$\bigoplus_{i \in I} F_i^{\oplus n_i'} \subseteq F'.$$

By the rank constraints, we conclude that $n_i' = n_i$, hence $F \simeq F'$. This prove the lemma. $\square$

[16]Note that the notion of semistability, $S$-equivalence and polystability are the same for either group $\mathrm{GL}(V)$ or $\mathrm{SL}(V)$. It is only the stability that depends on the choice.

**Theorem 54.** *Suppose that* $\gcd(r,d)=1$. *Then the functor* $\mathrm{pl.Coh}(X)^{ss}_{(r,d)}$ *is representable. In other words, there is a universal stable bundle* $\mathcal{F}$ *on* $M(r,d)\times X$.

*Proof.* Recall that the moduli space $M(r,d)$ is obtained as a GIT quotient of $\pi : R^{ss}\to M(r,d)$. Since $\gcd(r,d)=1$, we know that $\pi$ is actually a geometric quotient. Even better, all the stable points $x\in R^s=R^{ss}$ have a trivial stabilizer subgropu inside $\mathrm{PGL}(V)$. By Luna's etale slice theorem for GIT quotient, this implies that $\pi : R^{ss}\to M(r,d)$ is actually a principle $\mathrm{PGL}(V)$-bundle. In this case, theory of descent gives an equivalence of categories

$$(\pi\times \mathrm{id}_X)^* : \mathrm{Coh}(M(r,d)\times X)\xrightarrow{\sim}\mathrm{Coh}^{\mathrm{PGL}(V)}(R^{ss}\times X).$$

On the other hand, we had a universal quotient bundle $\widetilde{\mathcal{F}}$ over $R^{ss}\times X$ that is naturally a $\mathrm{GL}(V)$-equivariant. For $\widetilde{\mathcal{F}}$ to descend to a universal bundle on $M(r,d)\times X$, we need $\mathbb{C}^*$ to act trivially on $\widetilde{\mathcal{F}}$ which is generally not the case. Since $\mathbb{C}^*$ acts trivially on $R^{ss}\times X$, we have a weight decomposition

$$\widetilde{\mathcal{F}}=\bigoplus_{n\in\mathbb{Z}}\widetilde{\mathcal{F}}_n\mathbf{t}^n.$$

Since $\widetilde{\mathcal{F}}$ is a family of stable bundles, there can not be a non-trivial splitting and we have $\widetilde{\mathcal{F}}=\widetilde{\mathcal{F}}_n$ for some $n\in\mathbb{Z}$.

We now use $\widetilde{\mathcal{F}}$ to construct some line bundle on $R^{ss}$ of weight $-n$. Let $\mathcal{O}_X(1)$ be a degree 1 line bundle on $X$. Choose $m>>0$ so that $H^1(X,F(m))=0$ for all $F\in\mathrm{Coh}(X)^{ss}_{(r,d)}$. Then we have a rank $\chi(F(m))$ vector bundle

$$\mathcal{U}_m:=\widetilde{p}_*\Big(\widetilde{\mathcal{F}}\otimes\widetilde{q}^*\mathcal{O}_X(m)\Big)$$

where $\widetilde{p},\widetilde{q}$ denotes two projections from $R^{ss}\times X$. From construction, a vector bundle $\mathcal{U}_m$ has a natural $\mathrm{GL}(V)$-equivariant structure and its restriction to $\mathbb{C}^*\subseteq\mathrm{GL}(V)$ is of weight $n$. Taking a determinant $\det(\mathcal{U}_m)$, we obtain a line bundle on $R^{ss}$ of $\mathbb{C}^*$-weight $n\cdot\chi(F(m))$. Similarly, we have $\det(\mathcal{U}_{m+1})$ a line bundle of weight $n\cdot\chi(F(m+1))$. From the condition $\gcd(r,d)=1$, one can check that

$$\gcd(\chi(F(m)),\chi(F(m+1)))=\gcd(d+r(m+1-g),d+r(m+2-g))=1.$$

So there exists $a,b\in\mathbb{Z}$ such that

$$L:=\det(\mathcal{U}_m)^{\otimes a}\otimes\det(\mathcal{U}_{m+1})^{\otimes b}$$

is a line bundle on $R^{ss}$ of $\mathbb{C}^*$-weight $-n$. By the descent theory, $\widetilde{p}^*L\otimes\widetilde{\mathcal{F}}$ corresponds to a vector bundle $\mathcal{F}$ on $M(r,d)\times X$. From the construction, $\mathcal{F}$ has a property that for each $[F]\in M(r,d)$, we have $\mathcal{F}\Big|_{[F]}\simeq F$. We leave for the reader to check that $\mathcal{F}$ is indeed a universal family. $\square$

## 7. Deformation theory of sheaves

In the previous section, we have shown that $\mathrm{Coh}^s_{(r,d)} \subseteq \mathrm{Coh}^{ss}_{(r,d)}$ is corepresented by a (quasi)-projective scheme $M^s(r,d) \subseteq M(r,d)$. In fact, some general result about local property of the GIT quotient (called Luna's etale slice theorem) implies that a geometric quotient $R^s \to M^s(r,d)$ is a principle $\mathrm{PGL}(V)$-bundle.[17] In this section, we study local property of $M^s(r,d)$ via deformation theory and show that it is smooth of dimension

$$\mathrm{ext}^1(F,F) = 1 + r^2(g-1),$$

where $F$ is any stable bundle of rank $r$ and degree $d$. Strictly semistable locus $M(r,d) \backslash M^s(r,d)$, when it is non-empty, is exactly the singular locus of $M(r,d)$ unless we are in the case of $(g,r,d) = (2,2,\mathrm{even})$. We will not need this fact in the lecture.

7.1. **deformation functor.** We give a brief introduction to the deformation theory. For the interested reader for details, we refer to [H]. Let $M$ be any scheme of finite type over $\mathbb{C}$. Most basic local invariant of $M$ at a point $p \in |M|$ is given by the Zariski tangent space

$$T_{M,p} := \left(m_p/m_p^2\right)^\vee$$

which contains an infinitesimal deformation of the point $p \hookrightarrow M$. There is another local invariant of $M$ which contains significantly more data than $T_{M,p}$. Consider a $m_p$-adic completion $\widehat{\mathcal{O}}_{X,p}$ which is defined as an inverse limit of the system

$$\widehat{\mathcal{O}}_{X,p} := \varprojlim_{n\geq 1} \mathcal{O}_{X,p}/(m_p)^n.$$

This is again a local noetherian ring, though not finitely generated in general, with a unique maximal ideal $\widehat{m}_p$. We call $\left(\widehat{\mathcal{O}}_{X,p}, \widehat{m}_p\right)$ a *formal neighborhood* of a closed point $p$ in $M$. The formal neighborhood knows a great deal of the local property of $M$ at $p$. In particular it knows whether $M$ is smooth at $p$ or not which was not captured by $T_{M,p}$. Deformation theory is a strong technique to study the formal neighborhood of various moduli spaces.

We say that a $\mathbb{C}$-algebra $A$ is *Artin local ring* if it is a local ring which is finite dimensional $\mathbb{C}$-vector space. Let $\mathsf{Art}$ be a category of Artin local rings. For any stable bundle $F \in M^s(r,d)$, we define a *deformation functor* $D_F : \mathsf{Art} \to \mathsf{Set}$ where

$$D_F(A) := \left\{(\mathcal{F},\phi) \,\middle|\, \mathcal{F} :\ A\text{-flat family of bundles on } X,\ \phi : \mathcal{F} \otimes (A/m) \xrightarrow{\sim} F\right\}/\sim.$$

Equivalence relation $\sim$ is an isomorphism between $A$-flat family of sheaves that respect the isomorphism $\phi$ over the closed point. Note that $D_F$ is a functor because for every morphism $A' \to A$, we can pull back family of bundles from $A'$ to $A$. Roughly speaking, this is a local version of

[17] This is because stable points have a trivial stabilizer subgroup inside $\mathrm{PGL}(V)$.

the moduli functor $\mathrm{Coh}(X)^s_{(r,d)}$ around a point $F$. The lemma below is the precise version of this comment.

**Lemma 55.** *For each $A \in \mathsf{Art}$, we have an identification*

$$D_F(A) \xrightarrow{\sim} \mathrm{Mor}\Big(\big(\mathrm{Spec}(A), *\big), \big(M^s(r,d), [F]\big)\Big).$$

*Proof.* We first consider a map

$$D_F(A) \to \mathrm{Coh}(X)^s_{(r,d)}(\mathrm{Spec}(A))$$

which forgets a choice of an isomorphism $\phi : \mathcal{F} \otimes \mathbb{C} \to F$. This forgetful map is clearly a surjection. One can also show that this is injective because $F$ is a simple sheaf.

On the other hand, consider a diagram

$$R^s(\mathrm{Spec}(A))/\mathrm{PGL}(\mathrm{Spec}(A)) \to \mathrm{Coh}(X)^s_{(r,d)}(\mathrm{Spec}(A)) \to M^s(r,d)(\mathrm{Spec}(A)).$$

The first morphism is easily seen to be an isomorphism because $\mathrm{Spec}(A)$ is local. Also the composition of two morphisms is an isomorphism because $R^s \to M^s(r,d)$ is a principle $\mathrm{PGL}(V)$-bundle. Therefore the last morphism is an isomorphism proving the lemma. □

It is too much to expect for a deformation functor $D_F$ to be representable by some Artin local ring $R \in \mathsf{Art}$. Instead we consider a bigger category $\widehat{\mathsf{Art}}$ of Noetherian complete local $\mathbb{C}$-algebra with a residue field $\mathbb{C}$.

**Definition 56.** We say that a functor $D : \mathsf{Art} \to \mathsf{Set}$ is *pro-represented* by $R \in \widehat{\mathsf{Art}}$ if there are functorial isomorphisms

$$D_F(A) \xrightarrow{\sim} \mathrm{Mor}_{\widehat{\mathsf{Art}}}(R, A), \quad A \in \mathsf{Art}.$$

**Lemma 57.** *A functor $\mathsf{Art} \to \mathsf{Set}$ defined as $A \mapsto \mathrm{Mor}\big((\mathrm{Spec}(A), *), (M, p)\big)$ is pro-represented by a formal neighborhood of $p$ in $M$.*

*Proof.* We need to construct a functorial isomorphisms

$$\mathrm{Mor}\big((\mathrm{Spec}(A), *), (M, p)\big) \xrightarrow{\sim} \mathrm{Mor}_{\widehat{\mathsf{Art}}}\big(\widehat{\mathcal{O}}_{M,p}, A\big).$$

Consider a pointed morphism from $(\mathrm{Spec}(A), *)$ to $(M, p)$. This induces a morphism between local rings

$$\big(\mathcal{O}_{M,p}, m_p\big) \to (A, m).$$

Note that $m$-adic completion of $(A, m)$ is $A$ itself because $m^N = 0$ for $N >> 0$. Therefore, induced map on the completion is of the form

$$\widehat{\mathcal{O}}_{M,p} \to A.$$

Conversely, consider a morphism between local rings $\widehat{\mathcal{O}}_{M,p} \to A$. Choose $N >> 0$ such that $m^N = 0$. We define

$$\mathcal{O}_{M,p}/(m_p)^N \to A$$

by choosing any lift via a surjection $\widehat{\mathcal{O}}_{M,p} \twoheadrightarrow \mathcal{O}_{M,p}/(m_p)^N$. This is well-defined by our choice of $N$. This then defines a pointed morphism from $(\mathrm{Spec}(A), *)$ to $(M, p)$ via composition

$$\mathcal{O}_{M,p} \to \mathcal{O}_{M,p}/(m_p)^N \to A.$$

□

Therefore, we obtain the following.

**Corollary 58.** *Deformation functor $D_F$ is pro-represented by a formal neighborhood $\widehat{\mathcal{O}}_{M^s(r,d),[F]}$.*

Unlike the representability, pro-representing object is not necessarily unique. However, there are various properties of the pro-representing object that we can recover from the functor $D_F$, for example Zariski tangent space and smoothness.

**Lemma 59.** *If a functor $D : \mathsf{Art} \to \mathsf{Set}$ is pro-represented by $(R, m)$, then we have an identification*

$$D(\mathbb{C}[\epsilon]/\epsilon^2) \simeq T_{(R,m)}$$

*where the right hand side is a Zariski tangent space at a unique closed point.*

*Proof.* This follows from the sequence of identifications

$$D(\mathbb{C}[\epsilon]/\epsilon^2) \simeq \mathrm{Mor}_{\widehat{Art}}(R, \mathbb{C}[\epsilon]/\epsilon^2) \simeq \mathrm{Der}_{\mathbb{C}}(R, \mathbb{C}) =: T_{(R,m)}.$$

□

**Definition 60.** We say that a functor $D : \mathsf{Art} \to \mathsf{Set}$ has an *infinitesimal lifting property* if for every surjection $A' \twoheadrightarrow A$ in $\mathsf{Art}$ the induced map $D(A') \to D(A)$ is also surjective.

**Remark 61.** We say that a surjection $(A', m') \twoheadrightarrow (A, m)$ with a kernel $I$ is a *small extension* if $I \cdot m' = 0$. In such a case, we can regard the kernel $I$ as $A$-module. In many situations, it suffices to consider small extensions because any surjection $A' \twoheadrightarrow A$ can be factored into a composition of small extensions. Let $I$ be a kernel and $(m')^N = 0$. Then we have a factorization

$$A' = A'/I(m')^{N-1} \twoheadrightarrow A'/I(m')^{N-2} \twoheadrightarrow \cdots \twoheadrightarrow A'/I = A.$$

**Lemma 62.** *Let $D : \mathsf{Art} \to \mathsf{Set}$ be a functor pro-represented by $(R, m)$. Then $D$ has an infinitesimal lifting property if and only if $(R, m)$ is isomorphic to a formal power series ring over $\mathbb{C}$.*

*Proof.* If $(R, m)$ is isomorphic to a formal power series ring $\mathbb{C}[\![X_1, \dots, X_n]\!]$, then infinitesimal lifting property of $D$ follows from the universal property of a formal power series.

Conversely, assume that $D$ has an infinitesimal lifting property. Let $x_1, \dots, x_n \in R$ be elements whose image in the cotangent space $m/m^2$ form a basis over $\mathbb{C}$. Consider a morphism between short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}\langle x_1, \dots, x_n \rangle & \longrightarrow & \frac{\mathbb{C}[x_1,\dots,x_n]}{(x_ix_j)_{1\le i,j\le n}} & \longrightarrow & \frac{\mathbb{C}[x_1,\dots,x_n]}{(x_1,\dots,x_n)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & m/m^2 & \longrightarrow & R/m^2 & \longrightarrow & R/m & \longrightarrow & 0. \end{array}$$

Since the first and the third vertical arrows are isomorphism, the middle vertical arrow must be an isomorphism. Set $T := \mathbb{C}[\![X_1, \dots, X_n]\!]$ be a formal power series ring with a unique maximal ideal $m_T$. Similarly, we have

$$\mathbb{C}[X_1, \dots, X_n]/(X_iX_j)_{1\le i,j\le n} \xrightarrow{\sim} T/m_T^2.$$

Since $T$ is a complete local ring, we have an isomorphism

$$T = \varprojlim_{n\ge 2} T/m_T^n$$

where the inverse system consist of only surjections between Artin local rings. By the infinitesimal lifting property of $D = \mathrm{Mor}(R, -)$, a morphism

$$R \to R/m^2 \simeq T/m_T^2$$

admits a (non-unique) lifting $f : R \to T$ such that it induces an isomorphism between cotangent spaces

$$\overline{f} : m/m^2 \xrightarrow{\sim} m_T/m_T^2.$$

It is known that any morphism between complete local rings with a surjection on the cotangent space is a surjection.[18] Therefore $f : R \to T$ is a surjection. On the other hand, we may find a section $s : T \to R$ such that $f \circ s = \mathrm{id}$ by choosing a preimage $y_i \in f^{-1}(X_i)$. Since $s$ is a right inverse, it is necessarily injective. On the other hand, $s$ again induces an isormorphism on the cotangent space. This shows that both $s$ and $f$ are isomorphisms. □

7.2. **deformation of vector bundles.** Recall that we wanted to prove that $M^s(r, d)$ is smooth of dimension $\mathrm{ext}^1(F, F)$. From the results in the previous section, it suffices to prove that $D_F$ has an infinitesimal lifting property with $D_F(\mathbb{C}[\epsilon]/\epsilon^2) \simeq \mathrm{Ext}^1(F, F)$.

In this section, we prove a much more general result about a deformation of vector bundles. To state the main theorem, we need to introduce tangent-obstruction theory for the functor.

[18] See [Stacks project, 89.4.2]

**Definition 63.** Let $D : \mathsf{Art} \to \mathsf{Set}$ be a functor and $T$ and $O$ be a finite dimensional vector spaces. We say that $D$ admits a *tangent-obstruction theory* with respect to a tangent space $T$ and an obstruction space $O$ if the following conditions are satisfied. Let $A' \twoheadrightarrow A$ be any small extension with an ideal $I$. Then we have

(1) a $T \otimes_{\mathbb{C}} I$-action on $D(A')$,
(2) an obstruction map $\mathfrak{o} : D(A) \to O \otimes_{\mathbb{C}} I$

such that the following diagram is exact

$$T \otimes_{\mathbb{C}} I \curvearrowright D(A') \to D(A) \to O \otimes_{\mathbb{C}} I.$$

By exactness, we mean that image of $D(A') \to D(A)$ is exactly the preimage of $0 \in O \otimes_{\mathbb{C}} I$ and that every non-empty fiber of $D(A') \to D(A)$ is a torsor under $T \otimes_{\mathbb{C}} I$. We require these data to be functorial in certain sense for different choices of small extensions which we skip in this note.

**Theorem 64.** *Let $X$ be a smooth projective variety and $F$ be a vector bundle. The deformation functor $D_F$ admits a tangent-obstruction theory with a tangent space* $\mathrm{Ext}^1(F, F)$ *and an obstruction space* $\mathrm{Ext}^2(F, F)$.

*Proof.* We only sketch the proof. Let $A' \twoheadrightarrow A$ be a small extension with an ideal $I \subset A'$. We study when is the fiber of $D_F(A') \to D_F(A)$ non-empty and in such a case how does the fiber looks like.

Pick any element $(\mathcal{F}, \phi) \in D_F(A)$, i.e., a deformation of $F$ over $A$. We wish to construct a deformation $(\mathcal{F}', \phi') \in D_F(A')$ such that it restricts to $(\mathcal{F}, \phi)$ over $A$. Let $\{U_i\}$ be a Zariski open cover of $X$ such that a restriction $\mathcal{F}_i$ is free over $\mathrm{Spec}(A) \times U_i$ for every $i$. We would like to construct a vector bundle $\mathcal{F}'$ on $\mathrm{Spec}(A') \times X$ extending $\mathcal{F}$ that is also free over each $\mathrm{Spec}(A') \times U_i$. For that purpose, it suffices to choose a lifting of the gluing data $g_{ij} : \mathcal{F}_i\big|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j\big|_{U_{ij}}$ to

$$g'_{ij} : \mathcal{F}'_i\big|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}'_j\big|_{U_{ij}}$$

such that it satisfies the cocycle condition. We first choose any lifting of the gluing data $\{g'_{ij}\}$ which may not satisfies the cocycle condition. Define the composition

$$\delta_{ijk} := \left(g'_{ik}\right)^{-1} \circ g'_{jk} \circ g'_{ij}$$

which is an automorphism of a free vector bundle $\mathcal{F}'_i\big|_{U_{ijk}}$ that lifts an identity of $\mathcal{F}_i\big|_{U_{ijk}}$. By tensoring $\mathcal{F}'_i\big|_{U_{ijk}}$ to the exact sequence $0 \to I \to A' \to A \to 0$, we obtain

$$0 \to I \otimes F\big|_{U_{ijk}} \to \mathcal{F}'_i\big|_{U_{ijk}} \to \mathcal{F}_i\big|_{U_{ijk}} \to 0$$

since $I \cdot m' = 0$. Since $\delta_{ijk}$ lift the identity, we obtain a morphism

$$\delta_{ijk} - \mathrm{id} \in \mathrm{Hom}(\mathcal{F}'_i\big|_{U_{ijk}}, I \otimes F_i\big|_{U_{ijk}}) = \mathrm{Hom}(F\big|_{U_{ijk}}, F\big|_{U_{ijk}}) \otimes I$$

where the second equality is again because of $I \cdot m' = 0$. By computation, one can check that this satisfies a 2-cocycle condition hence defining an element in the cohomology

$$\mathfrak{o}((\mathcal{F}, \phi)) := \{\delta_{ijk} - \mathrm{id}\} \in \mathrm{Ext}^2(F, F) \otimes I.$$

One can also check that this class depends only on $(\mathcal{F}, \phi)$ and not on the choice of lifting $\{g'_{ij}\}$ nor the choice of a open covering $\{U_i\}$. This construction provides an obstruction map

$$\mathfrak{o} : D(A) \to \mathrm{Ext}^2(F, F) \otimes I.$$

If $\mathfrak{o}(\mathcal{F})$ vanishes, then we may choose a new choice of a lifting $\{g'_{ij}\}$ so that it satisfies the cocycle condition. This in turn defines a vector bundle $\mathcal{F}'$ over $\mathrm{Spec}(A') \times X$ that extends $\mathcal{F}$. Conversely, if $(\mathcal{F}, \phi) \in D(A)$ was in the image of $D(A')$ then $\mathfrak{o}(\mathcal{F})$ clearly vanishes. This proves the obstruction part of the theorem.

Now we turn our attention to the tangent space. Suppose that $\mathfrak{o}(\mathcal{F}, \phi) = 0$. Let $(\mathcal{F}', \phi')$ and $(\mathcal{F}'', \phi'')$ be any two lifting in $D(A')$. Let $\{U_i\}$ be an open covering such that both $\mathcal{F}'$ and $\mathcal{F}''$ are free over $\mathrm{Spec}(A') \times U_i$. Then we choose an isomorphism

$$g_i : \mathcal{F}'\big|_{U_i} \xrightarrow{\sim} \mathcal{F}''\big|_{U_i}$$

On the intersection $U_{ij}$ we obtain $\delta_{ij} := g_j^{-1} \circ g_i$ which is an automorphism of $\mathcal{F}'\big|_{U_{ij}}$ lifting the identity map for $\mathcal{F}\big|_{U_{ij}}$. By the same method as in the previous paragraph, this defines a 1-cocycle

$$\{\delta_{ij} - \mathrm{id}\} \in \mathrm{Ext}^1(F, F) \otimes I.$$

One can check that this class is zero if and only if we can adjust the choice of $\{g_i\}$ in a way that it glues to an isomorphism $g : \mathcal{F}' \to \mathcal{F}''$ which restricts to an identity over $A$. Therefore, one we fix an element $(\mathcal{F}', \phi') \in D(A')$ in the fiber over $(\mathcal{F}, \phi)$, any other extensions are in bijection with $\mathrm{Ext}^1(F, F) \otimes I$. This defines a desired torsor structure. □

**Corollary 65.** *Let $X$ be a smooth projective curve of genus $g$. Let $M^s(r, d)$ be a moduli space of stable bundles of rank $r$ and degree $d$. Then $M^s(r, d)$ is smooth of dimension $1 + r^2(g-1)$ if it is non-empty. Moreover, we can canonically identify the tangent space at $[F] \in M^s(r, d)$ with $\mathrm{Ext}^1(F, F)$.*

*Proof.* Let $[F] \in M^s(r, d)$. We have seen that a deformation functor $D_F$ is pro-represented by a formal neighborhood $\widehat{\mathcal{O}}_{M^s(r,d),[F]}$. On the other hand, previous theorem implies that $D_F$ admits a tangent-obstruction theory with a tangent space $\mathrm{Ext}^1(F, F)$ and an obstruction space $\mathrm{Ext}^2(F, F) = 0$. Since there is no obstruction, $D_F$ has an infinitesimal lifting property. Therefore a formal neighborhood is a power series ring hence, $M^s(r, d)$ is smooth at a point $[F]$ via formal criterion of smoothness. Consider a small extension $\mathbb{C}[\epsilon]/\epsilon \twoheadrightarrow \mathbb{C}$. Then $D_F(\mathbb{C}[\epsilon]/\epsilon^2)$ is a torsor over

$\mathrm{Ext}^1(F,F)$. Since $D_F(\mathbb{C}[\epsilon]/\epsilon^2)$ has a canonical element that corresponds to a trivial deformation, this canonically identifies

$$D_F(\mathbb{C}[\epsilon]/\epsilon^2) \simeq \mathrm{Ext}^1(F,F).$$

Since $F$ is stable hence simple, we know that $\hom(F,F) = 1$. By Riemann-Roch formula, we obtain

$$\mathrm{ext}^1(F,F) = 1 - \chi(F,F) = 1 + r^2(g-1).$$

□

7.3. **determinant morphism and trace map.** Consider $\mathrm{Pic}^d(X) := \mathrm{Coh}(X)^{ss}_{(1,d)}$ a functor for degree $d$ line bundles which are automatically stable. We have seen that this is represented by a moduli space $\mathsf{Pic}^d_X := M_X(1,d)$ which we call a Picard variety for degree $d$ line bundles. By Corollary 65, Picard variety is smooth of dimension $g$. We have $\mathsf{Pic}^d_X \simeq \mathsf{Pic}^{d'}_X$ for any $d, d' \in \mathbb{Z}$ by tensoring with some line bundle of degree $d' - d$. Picard variety $\mathsf{Pic}^0_X$ is of special interest because it has a natural group structure by a tensor product of degree 0 line bundles.

So far, we have been considering the moduli space $M(r,d)$ of semistable bundles with rank $r$ and degree $d$. There is a variation of this moduli space where we fix the determinant of the semistable bundles. Recall that we have a diagram between functors

$$\begin{tikzcd} \mathrm{Coh}(X)^{ss}_{(r,d)} \arrow[r] \arrow[d, "\det"'] & M_X(r,d) \arrow[d, dashed, "\det"] \\ \mathrm{Pic}^d(X) \arrow[r] & \mathsf{Pic}^d_X \end{tikzcd}$$

where the horizontal arrows are corespresenting morphisms and the vertical arrows are determinant morphisms. To be more precise, we define the left column by sending an $S$-flat family $\mathcal{F}$ of semistable bundles of rank $r$ and degree $d$ to a $S$-flat family $\det(\mathcal{F})$ of line bundles of degree $d$ on $S \times X$. This then induces a morphism between moduli spaces due to universal property. Let $L$ be any line bundle of degree $d$. Then we define a moduli space $M_X(r,L)$ of semistable bundles of rank $r$ and determinant $L$ as a fiber product

$$\begin{tikzcd} M_X(r,L) \arrow[r, hook] \arrow[d] & M_X(r,d) \arrow[d, "\det"] \\ \{L\} \arrow[r, hook] & \mathsf{Pic}^d_X. \end{tikzcd}$$

One can also define $M_X(r,L)$ as a scheme that corepresents the appropriate functor.

For now let $X$ be a smooth projective variety, not necessarily a curve, and $F$ be a vector bundle over $X$ with $\det(F) = L$. We have a trace morphism between vector bundles

$$\mathrm{tr} : \mathcal{H}om(F,F) \to \mathcal{O}_X$$

and the constant multiplication map

$$\mathrm{id} : \mathcal{O}_X \to \mathcal{H}\mathrm{om}(F, F)$$

such that $\mathrm{tr} \circ \mathrm{id}$ is a constant multiplication map by $r = \mathrm{rk}(F)$. These maps induce morphisms in cohomology groups and the induced trace maps between cohomology groups

$$\mathrm{tr}_i : \mathrm{Ext}^i(F, F) \to H^i(\mathcal{O}_X), \quad \mathrm{id}_i : H^i(\mathcal{O}_X) \to \mathrm{Ext}^i(F, F).$$

Since $\mathrm{tr}_i \circ \mathrm{id}_i$ is a constant multiplication by $r > 0$, $\mathrm{tr}_i$ must be a surjection whose kernel we denote by $\mathrm{Ext}^i(F, F)_0$. We have a splitting exact sequence

$$0 \to \mathrm{Ext}^i(F, F)_0 \to \mathrm{Ext}^i(F, F) \to H^i(\mathcal{O}_X) \to 0.$$

It turns out that trace morphism $\mathrm{tr}_i$ when $i = 1, 2$ is closely related to the determinant morphism via deformation theory. Precise relation is as follows. Let $D_F$ and $D_L$ be a deformation functor for a vector bundle and a line bundle, respectively. We have a determinant morphism $\det : D_F \to D_L$ between these functors. From Theorem 64, $D_F$ and $D_L$ admit a tangent-obstruction theory with $\mathrm{Ext}^i(F, F)$ and $\mathrm{Ext}^i(L, L) = H^i(\mathcal{O}_X)$ with $i = 1, 2$, respectively. Tangent-obstruction theory of $D_F$ and $D_L$ are compatible with respect to the determinant morphism $\det : D_F \to D_L$ in the sense that

$$\left(\mathrm{tr}_2 \otimes \mathrm{id}_I\right) \mathfrak{o}(\mathcal{F}, \phi) = \mathfrak{o}(\det(\mathcal{F}), \det(\phi)).$$

One has a similar compatibility for the torsor structure by the tangent theory as well whose exact formulation we omit. Therefore, the deformation theory of a determinant fixed obstruction theory has a tangent-obstruction theory with $\mathrm{Ext}^i(F, F)_0$ with $i = 1, 2$.

**Proposition 66.** *Let $X$ be a smooth projective curve of genus $g$. Let $M^s(r, L)$ be a moduli space of stable bundles of rank $r$ and determinant $L$. Then $M^s(r, L)$ is smooth of dimension $(r^2 - 1)(g - 1)$ if it is non-empty. Moreover, we can canonically identify the tangent space at $[F] \in M^s(r, L)$ with $\mathrm{Ext}^1(F, F)_0$.*

7.4. **deformation of quotients.** In this section, we record a fact about deformation theory of quotients. Let $X$ be a smooth projective variety and $V$ be a fixed coherent sheaf on $X$. Define a deformation functor $D_{[V \twoheadrightarrow F]} : \mathsf{Art} \to \mathsf{Set}$ for a quotient as $D_{[V \twoheadrightarrow F]}(A) = \left\{([V_A \twoheadrightarrow \mathcal{F}], \phi)\right\}/\sim$ where $[V_A \twoheadrightarrow \mathcal{F}]$ is an $A$-flat family of quotients of $V$ and $\phi$ is an isomorphism between

$$\phi : [V_A \twoheadrightarrow \mathcal{F}] \otimes (A/m) \xrightarrow{\sim} [V \twoheadrightarrow F].$$

Since Quot scheme is a representable functor, studying the deformation functor would let us study tangent space and smoothness criterion. The following is theorem is a quotient version of Theorem 64.

**Theorem 67.** *The deformation functor $D_{[V\twoheadrightarrow F]}$ admits a tangent-obstruction theory with a tangent space* $\mathrm{Hom}(K,F)$ *and an obstruction space* $\mathrm{Ext}^1(K,F)$ *where* $K:=\ker(V\twoheadrightarrow F)$.

*Proof.* See [H] for the proof. □

**Corollary 68.** *Moduli space $M(r,d)$ is a locally normal variety.*[19]

*Proof.* Since good quotient (in particular categorical quotient) preserves local normality, it suffices to show that $R^{ss}$ is locally normal. We will in fact show that $R^{ss}$ is smooth using the deformation theory of quotient. Recall that $R^{ss}$ is an open set of $\mathsf{Quot}_X(V\otimes\mathcal{O}_X,(r,d))$ that corresponds to a quotient $[V\otimes\mathcal{O}_X\twoheadrightarrow F]$ with $F$ a semistable bundle of rank $r$ degree $d$. Therefore, it suffices to show that

$$\mathrm{Ext}^1(K,F)=0$$

where $K=\ker(V\otimes\mathcal{O}_X\twoheadrightarrow F)$. By taking $\mathrm{Hom}(-,F)$ to the exact sequence

$$0\to K\to V\otimes\mathcal{O}_X\to F\to 0,$$

we obtain

$$\cdots\to\mathrm{Ext}^1(V\otimes\mathcal{O}_X,F)\to\mathrm{Ext}^1(K,F)\to\mathrm{Ext}^2(F,F)\to\cdots.$$

Since we assume $d>>0$ in the construction, we have $\mathrm{Ext}^1(V\otimes\mathcal{O}_X,F)=0$ from semistability of $F$. Also $\mathrm{Ext}^2(F,F)=0$ for dimension reason. Therefore, we prove that obstruction space vanishes hence $R^{ss}$ is smooth of dimension $\chi(K,F)$. □

## 8. Geometric properties of moduli spaces

In this section, we study several geometric properties of moduli spaces.

8.1. **non-emptyness.** In the previous section, we have proven that $M^s(r,d)$ is smooth of dimension $1+r^2(g-1)$ if it is non-empty. To get rid of this assumption, we give a criterion for a existence of stable bundles. The case of $g=0$ or $g=1$ is somewhat exceptional so it requires a separate treatment. We only record the main results in this direction without proof which are due to Grothendieck for $\mathbb{P}^1$ case and Atiyah for elliptic curve case.

**Theorem 69.** *Let $X$ be a smooth projective curve of genus 0, i.e., $X\simeq\mathbb{P}^1$. Every vector bundle on $X$ is of the form*

$$V\simeq\mathcal{O}_{\mathbb{P}^1}(d_1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1}(d_r)$$

*where $d_1\geq\cdots\geq d_n$. In particular, there are no stable bundles of rank at least 2.*

[19] We say $M$ is a locally normal if each stalk $\mathcal{O}_{M,x}$ is a normal domain. For Noetherian schemes, this condition is equivalent to being a disjoint union of a normal varieties.

**Theorem 70.** *Let $X$ be a smooth projective curve of genus 1, i.e., an elliptic curve. There exists a stable bundle of degree $(r,d)$ if and only if $\gcd(r,d)=1$. Moreover, in such a case, there exists a unique stable bundle of determinant $L$ for any fixed line bundle $L$ of degree $d$.*

Now we state the main existence result for generic case of genus at least two.

**Theorem 71.** *Let $X$ be a smooth projective curve of genus at least 2. There exists a stable bundle of any rank and degree.*

*Proof.* Let $(r,d)$ be any pair of integers with $r \geq 1$. We may assume that $d$ is arbitrarily large. Then there exists a vector bundle $F$ of rank $r$ and degree $d$ such that it is globally generated and $H^1(X,F)=0$. Let $V$ be a vector space of dimension $\chi(F)=d+r(1-g)$. Consider a quot scheme

$$\mathsf{Quot} := \mathsf{Quot}_X(V \otimes \mathcal{O}_X, (r,d)).$$

We study the Quot scheme near the point

$$p = [V \otimes \mathcal{O}_X \twoheadrightarrow F] \in \mathsf{Quot}$$

where it induces an isomorphism $V \xrightarrow{\sim} H^0(X,F)$ on the global sections. We first show that the Quot scheme is smooth at a point $p$. Consider the exact sequence

$$0 \to K \to V \otimes \mathcal{O}_X \to F \to 0.$$

By Theorem 67, we know that a deformation functor $D_{[V\otimes\mathcal{O}_X \twoheadrightarrow F]}$ admits a tangent-obstruction theory with a tangent space $\mathrm{Hom}(K,F)$ and an obstruction space $\mathrm{Ext}^1(K,F)$. On the other hand, by taking $\mathrm{Hom}(-,F)$ to the previous exact sequence, we obtain the long exact sequence whose part of it looks like

$$\cdots \to \mathrm{Ext}^1(V\otimes\mathcal{O}_X,F) \to \mathrm{Ext}^1(K,F) \to \mathrm{Ext}^2(F,F) \to \cdots.$$

Since $H^1(X,F)=0$ by assumption and $\mathrm{Ext}^2(F,F)=0$ for dimension reason, we obtain $\mathrm{Ext}^1(K,F)=0$. Since obstruction space vanishes, the $p \in \mathsf{Quot}$ is a smooth point of dimension $\chi(K,F)$.

Let $S \subseteq \mathsf{Quot}$ be a smooth irreducible affine neighborhood of $p$ and

$$V \otimes \mathcal{O}_{S\times X} \twoheadrightarrow \mathcal{F}$$

be the universal quotient over $S \times X$. We may assume that $\mathcal{F}$ is a family of vector bundles with vanishing higher cohomologies by shrinking $S$ if necessary. We claim that there is a closed point $s \in |S|$ such that $F_s$ is a stable bundle. For this purpose, we study the relative quot scheme

$$\pi_{(r'',d'')} : \mathsf{Quot}_{S\times X/S}(\mathcal{F},(r'',d'')) \to S$$

for each $(r'', d'')$ with $0 < r'' < r$ and $\mu \geq \mu''$. Since quotient of $\mathcal{F}$ also gives a quotient of $V \otimes \mathcal{O}_{S\times X}$, such a Quot scheme is empty unless $0 \leq \mu''$. Therefore it suffices to consider finitely many such $(r'', d'') \in I$ and prove that

$$S^s := S \setminus \bigcup_{(r'',d'')\in I} \operatorname{image}\Big(\mathsf{Quot}_{S\times X/S}(\mathcal{F}, (r'', d'')) \to S\Big)$$

is non-empty. Note that each image is a closed subset and there are only finitely many indexes in $I$. Therefore it suffices to prove that for each fixed $(r'', d'') \in I$, the image of $\pi = \pi_{(r'',d'')}$ is not entire $S$.

Denote the relative universal quotient as

$$0 \to \mathcal{F}' \to \widetilde{q}^*\mathcal{F} \to \mathcal{F}'' \to 0$$

which is define over

$$\mathsf{Quot}_{S\times X/S}(\mathcal{F}, (r'', d'')) \times_S (S \times X)$$

with $\widetilde{p}, \widetilde{q}$ denoting the two projections. Let $x$ be a closed point of the relative Quot scheme and let $s = \pi(x) \in |S|$. Then $x$ represents a certain quotient $0 \to F' \to F_s \to F'' \to 0$ with $\mathrm{ch}(F'') = (r'', d'')$. Consider a differential

$$d\pi\big|_x : T_x\Big(\mathsf{Quot}_{S\times X/S}(\mathcal{F}, (r'', d''))\Big) \to T_s(S).$$

The kernel of the differential $d\pi\big|_x$ represents a tangent space of a non-relative Quot scheme $\mathsf{Quot}_X(F_s, (r'', d''))$ at a point $x$. By the deformation theory of a quotient, this is identified with $\mathrm{Hom}(F', F'')$. In other words, we have an exact sequence

$$0 \to \mathrm{Hom}(F', F'') \to T_x\Big(\mathsf{Quot}_{S\times X/S}(\mathcal{F}, (r'', d''))\Big) \to T_s(S).$$

We further claim that this exact sequence can be extended to a longer exact sequence

$$0 \to \mathrm{Hom}(F', F'') \to T_x\Big(\mathsf{Quot}_{S\times X/S}(\mathcal{F}, (r'', d''))\Big) \to T_s(S) \to \mathrm{Ext}^1(F', F'') \tag{2}$$

where we call the last map a Kodaira-Spencer map. The Kodaira-Spencer map is constructed as follows. Let $v \in T_s(S)$ be a tangent vector that represents a pointed morphism

$$v : \Big(\mathrm{Spec}\big(\mathbb{C}[\epsilon]/\epsilon^2\big), *\Big) \to \big(S, s\big).$$

Consider the following pull back diagram via $v$:

$$\begin{array}{ccccc} F_s & \longrightarrow & \mathcal{F}_v & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}\big(\mathbb{C}[\epsilon]/\epsilon^2\big) \times X & \longrightarrow & S \times X. \end{array}$$

Note that we are given a point $x$ that represents a quotient $0 \to F' \to F_s \to F'' \to 0$. By deformation theory of a quotient, we have an obstruction class

$$\mathfrak{o}([F_s \twoheadrightarrow F'']) \in \mathrm{Ext}^1(F', F'')$$

defining the image of $v \in T_s(S)$ under the Kodaira-Spencer map. Moreover, since $[F_s \twoheadrightarrow F'']$ can be extended to a small extension by $v$ if and only if the obstruction class vanishes, we also have the exactness of the sequence (2).

We also claim that Kodaira-Spencer map is surjective. This is because we can factorize the Kodaira-Spencer map as

$$T_s(S) = \mathrm{Hom}(K_s, F_s) \xrightarrow{f} \mathrm{Ext}^1(F_s, F_s) \xrightarrow{g} \mathrm{Ext}^1(F', F_s) \xrightarrow{h} \mathrm{Ext}^1(F', F'').$$

See proof of [L, Theorem 8.6.1] for details for this factorization. Morphisms $f, g$ and $h$ are surjective because of the vanishings

$$\mathrm{Ext}^1(V \otimes \mathcal{O}_X, F_s) = 0, \quad \mathrm{Ext}^2(F'', F_s) = 0, \quad \mathrm{Ext}^2(F', F') = 0.$$

To show that the morphism $\pi = \pi_{(r'', d'')}$ is not surjective, it suffices to prove that $d\pi\big|_x$ is not surjective. Or equivalently, we must show that $\mathrm{Ext}^1(F', F'') \neq 0$. This follows from the computation

$$\begin{aligned} \mathrm{ext}^1(F', F'') &\geq -\chi(F', F'') \\ &= r'r''(\mu' - \mu'' + g - 1) \\ &> 0. \end{aligned}$$

In the last inequality, we used the that $g \geq 2$ and $\mu' \geq \mu \geq \mu''$. Note that the same strict inequality hold if $g = 1$ and $\mu' > \mu > \mu''$ proving the non-emptiness of the moduli of semistable bundles on elliptic curves. □

### 8.2. **irreducibility and unirationality.**

**Proposition 72.** *The Picard variety* $\mathsf{Pic}^d(X)$ *is smooth projective variety of dimension* $g = g(X)$.

*Proof.* From deformation theory, we know that $\mathsf{Pic}^d(X)$ is smooth of dimension $g$. Therefore, it suffices to prove the irreducibility. We may and will assume that $d \geq g$. Consider a symmetric product $\mathrm{Sym}^d(X)$, which can also be considered as a Hilbert scheme parametrizing zero dimensional subschemes of length $d$. As a Hilbert scheme, it is equipped with a universal object

$$0 \to I_{\mathcal{D}} \to \mathcal{O}_{\mathrm{Sym}^d(X) \times X} \to \mathcal{O}_{\mathcal{D}} \to 0.$$

If we consider the line bundle $\det(\mathcal{O}_{\mathcal{D}})$ as a family of degree $d$ line bundle parametrized by $\mathrm{Sym}^d(X)$, this induces a morphism

$$\pi : \mathrm{Sym}^d(X) \to \mathsf{Pic}^d(X)$$

since $\mathsf{Pic}^d(X)$ is a coarse moduli space. Since $\mathrm{Sym}^d(X)$ is irreducible, it suffices to prove that $\pi$ is surjective. Note that for every line bundle $L \in \mathsf{Pic}^d(X)$, the fiber of $\pi$ is $\mathbb{P}(H^0(X,L))$ because this corresponds to a degree $d$ effective divisor $D$ such that $\det(\mathcal{O}_D) = \mathcal{O}_X(D) \simeq L$. Since $d \geq g$, we know that $H^1(X,L) = 0$ hence $h^0(X,L) = \chi(L) = d + (1-g) \geq 1$. This proves that $\pi$ is surjective. ☐

**Proposition 73.** *The moduli space $M(r,L)$ is an unirational variety if it is non-empty.*

*Proof.* We may assume that $d > d(g,r)$. Suppose for the moment that every $[F] \in M(r,L)$ is obtained as a certain extension

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to F \to L \to 0$$

which is parametrized by $W := \mathrm{Ext}^1(L, \mathcal{O}_X^{\oplus(r-1)})$. Since $\mathrm{Hom}(L, \mathcal{O}_X^{\oplus(r-1)}) = 0$ if $d > 0$, we know that $W$ is a vector space of dimension $-(r-1)\cdot\chi(L,\mathcal{O}_X) = (r-1)(d+g-1)$. We also have a universal extension

$$0 \to q^*\mathcal{O}_X^{\oplus(r-1)} \to \mathcal{F} \to q^*L \to 0$$

over $W \times X$. Since semistability is an open condition, we have a corresponding open subset $W^{ss} \subseteq W$. If every $[F] \in M(r,L)$ arises as an extension of this kind, then the moduli map $W^{ss} \to M(r,d)$ is surjective proving that $M(r,L)$ is an unirational variety.

Now we prove the previous claim. Let $F$ be a semistable bundle of rank $r$ and determinant $L$ of degree $d$. By choice of $d > d(g,r)$, we know that $F$ is globally generated. Consider a Grassmannian

$$\mathrm{Gr} := \left\{ V \subseteq H^0(X,F) \,\middle|\, \dim(V) = (r-1) \right\}.$$

We claim that there is $[V] \in \mathrm{Gr}$ such that the evaluation map

$$V \otimes \mathcal{O}_X \to F$$

detfines a rank $(r-1)$ subbundle. We do the dimension count. Each point $x \in |X|$ defines a subspace

$$W_x := \ker\left( H^0(X,F) \twoheadrightarrow F\big|_x \right)$$

of codimension $r$. Then $W_x$ defines a Schubert cycle

$$\sigma_x := \left\{ V \subseteq H^0(X,F) \,\middle|\, V \cap W_x \neq 0 \right\}$$

of codimension 2. Varying $x \in |X|$, we conclude that $\cup_{x\in|X|}\sigma_x$ has codimension at least one in Gr. If we choose $[V] \in \mathrm{Gr}$ that avoids this locus, then the evaluation map defines a subbundle. Since the quotient of $V \otimes \mathcal{O}_X \to F$ is a line bundle of determinant $L$, this proves the claim. ☐

**Corollary 74.** *The moduli space $M(r,d)$ is irreducible if non-empty.*

*Proof.* We can use the determinant morphism $M(r,d) \to \mathsf{Pic}^d(X)$ and the above argument. We leave the detail for the reader. $\square$

## 9. Tautological classes and relations

In this section, we restrict to the cases with coprime rank $r > 0$ and degree $d$. By deformation theory, we have proven that the moduli space $M(r,d)$ is a smooth variety of dimension $1+r^2(g-1)$. Now we would like to study some global property of the moduli space $M(r,d)$. Similar to the case of Grassmannian, we use tautological classes and tautological relations to study the cohomology group $H^*(M(r,d),\mathbb{Q})$. This requires existence of a universal bundle which is one of the reason why we restrict to the case with $\gcd(r,d) = 1$.

9.1. **Tautological classes.** Consider a moduli space $M = M_X(r,d)$ of stable bundles of rank $r$ and degree $d$ on a smooth projective curve $X$ with $\gcd(r,d) = 1$. There exist a universal bundle $\mathcal{F}$ on $M \times X$ which is unique up to ambiguity of $\mathsf{Pic}(M)$. Using the universal bundle $\mathcal{F}$, we can construct various cohomology classes on the moduli space $M$.

**Definition 75.** For each $k \geq 0$ and $\gamma \in H^*(X,\mathbb{Q})$, define a class

$$\mathrm{ch}_k(\gamma) := p_*\Big(\mathrm{ch}_k(\mathcal{F}) \cup q^*(\gamma)\Big) \in H^*(M,\mathbb{Q}).$$

By a *tautological class* of $M(r,d)$ (with respect to a choice of a universal bundle $\mathcal{F}$), we mean a polynomial in the classes of the form $\mathrm{ch}_k(\gamma)$ for various $k$'s and $\gamma$'s.

**Remark 76.** For notational simplicity, we omitted the dependence of $\mathrm{ch}_k(\gamma)$ on the choice of $\mathcal{F}$. However, we can show that certain tautological classes do not depend on the choice of universal bundle.

**Remark 77.** We may also use chern classes $c_k(\mathcal{F})$ instead of chern characters $\mathrm{ch}_k(\mathcal{F})$. Two approaches are equivalent by the following universal relation between chern classes and chern characters

$$1 + c_1 + c_2 + \cdots = \exp\left(\sum_{k\geq 1}(-1)^{k-1}(k-1)!\mathrm{ch}_k\right).$$

9.2. **Generators of the cohomology group.** Just like the case of Grassmannian, tautological classes generates the entire cohomology of the moduli space $M = M_X(r,d)$.

**Theorem 78.** *Tautological classes generate $H^*(M,\mathbb{Q})$ as a ring.*

*Proof.* We use so called Beauville's diagonal trick. Consider a product $M \times M \times X$ and two pull backs of a universal bundle

$$\mathcal{F}_1 := \pi_{13}^*\mathcal{F}, \quad \mathcal{F}_2 := \pi_{23}^*\mathcal{F}.$$

Denote a projection $\pi = \pi_{12} : M \times M \times X \to M \times M$ and consider a derived relative Hom complex

$$\mathrm{R}\mathcal{H}\mathrm{om}_\pi(\mathcal{F}_1, \mathcal{F}_2) \in D^b(M \times M).$$

Since $\pi$ is a smooth morphism of dimension 1, the derived relative Hom admits a quasi-isomorphism

$$\mathrm{R}\mathcal{H}\mathrm{om}_\pi(\mathcal{F}_1, \mathcal{F}_2) \xrightarrow{\sim} \big[E_0 \xrightarrow{\sigma} E_1\big].$$

where $E_i$ is a vector bundle of rank $r_i$. Note that for every point $p = (F_1, F_2) \in M \times M$, we have

$$\mathrm{RHom}(F_1, F_2) \xrightarrow{\sim} \left[E_0\Big|_p \xrightarrow{\sigma|_p} E_1\Big|_p\right]$$

by base change. On the other hand, since $F_1$ and $F_2$ are stable bundles of same slope, we have

$$\ker(\sigma|_p) = \mathrm{Hom}(F_1, F_2) = \begin{cases} \mathbb{C}, & \text{if } \ F_1 \simeq F_2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, morphism $\sigma : E_0 \to E_1$ is generically of rank $r_0$ which degenerates (at least set theoretically) over the diagonal $M \hookrightarrow M \times M$ where the rank drops exactly by 1. Denote the degeneracy locus of $\sigma$ by $Z$ whose scheme structure is defined as a zero locus of the morphism

$$\wedge^{r_0}\sigma : \wedge^{r_0} E_0 \to \wedge^{r_0} E_1.$$

By definition of the degeneracy loci $Z$, the kernel

$$\ker\left(E_0\Big|_Z \to E_1\Big|_Z\right)$$

is a line bundle, say $L$. Consider a product $Z \times X$ which two projections $p$ and $q$. Over $Z \times X$, we have $\mathcal{H}\mathrm{om}(\mathcal{F}_1\big|_Z, \mathcal{F}_2\big|_Z)$ whose pushforward along $p$ is given by

$$p_*\mathcal{H}\mathrm{om}(\mathcal{F}_1\big|_{Z\times X}, \mathcal{F}_2\big|_{Z\times X}) = L.$$

We have an adjunction morphism

$$\mathcal{F}_1\big|_{Z\times X} \to \mathcal{F}_2\big|_{Z\times X} \otimes p^*L$$

which restricts to non-zero morphism at each fibers hence necessarily an isomorphism. By universal property of the moduli space $M$, this implies that $Z$ factors through a diagonal $M \hookrightarrow M \times M$. Since $Z$ contains a diagonal, this proves that the degeneracy $Z$ is indeed scheme theoretically a diagonal.

From the above discussion, we have proven that diagonal $\Delta : M \hookrightarrow M \times M$ is degeneracy locus of a morphism $\sigma : E_0 \to E_1$. Recall that expected codimension of the degeneracy locus is given by

$$(r_0 - k)(r_1 - k)$$

where $k$ is the degenerate rank which is in our case $k = r_0 - 1$. Therefore, we have

$$(r_0 - k)(r_1 - k) = r_1 - r_0 + 1 = 1 - \chi(F_1, F_2) = \dim(M),$$

hence the diagonal is a degeneracy locus of expected codimension. In such a case, Thom-Porteus formula expresses the push forward class of the degeneracy locus as

$$\Delta_*[M] = c_d(E_1 - E_0), \quad d = \dim(M).$$

In particular, diagonal class depends only on the difference $E_1 - E_0 = -\mathrm{R}\mathcal{H}\mathrm{om}_\pi(\mathcal{F}_1, \mathcal{F}_2)$ rather than each $E_1$ and $E_0$. In sum up, we have

$$\Delta_*[M] = c_d\Big( - \mathrm{R}\mathcal{H}\mathrm{om}_\pi(\mathcal{F}_1, \mathcal{F}_2)\Big) \in H^*(M, \mathbb{Q}) \otimes H^*(M, \mathbb{Q}).$$

On the other hand, Grothendieck-Riemann-Roch formula gives

$$\mathrm{ch}\Big(\mathrm{R}\mathcal{H}\mathrm{om}_\pi(\mathcal{F}_1, \mathcal{F}_2)\Big) = \pi_{12*}\Big(\pi_{13}^*\mathrm{ch}^\vee\mathcal{F} \cdot \pi_{23}^*\mathrm{ch}\mathcal{F} \cdot \pi_3^*\mathrm{td}(X)\Big).$$

Let $\Delta_*\mathrm{td}(X) = \sum_{i\in I} \alpha_i^L \otimes \alpha_i^R \in H^*(X, \mathbb{Q}) \otimes H^*(X, \mathbb{Q})$. Decomposing the morphism $\pi_{12}$ as

$$\pi_{12} : M \times M \times X \xrightarrow{\mathrm{id}_{M\times M} \times \Delta_X} M \times M \times X \times X \xrightarrow{p_{12}} M \times M,$$

we can rewrite the above formula as

$$\begin{aligned}
\mathrm{ch}\Big(\mathrm{R}\mathcal{H}\mathrm{om}_\pi(\mathcal{F}_1, \mathcal{F}_2)\Big) &= p_{12*}\Big(p_{13}^*\mathrm{ch}^\vee\mathcal{F} \cdot p_{23}^*\mathrm{ch}\mathcal{F} \cdot p_{34}^*\big(\Delta_*\mathrm{td}(X)\big)\Big) \\
&= \sum_{i\in I} p_{12*}\Big(p_{13}^*\mathrm{ch}^\vee\mathcal{F} \cdot p_{23}^*\mathrm{ch}\mathcal{F} \cdot p_3^*\alpha_i^L \cdot p_4^*\alpha_i^R\Big) \\
&= \sum_{i\in I} \Big[p_*\big(\mathrm{ch}^\vee\mathcal{F} \cdot q^*\alpha_i^L\big)\Big] \otimes \Big[p_*\big(\mathrm{ch}\mathcal{F} \cdot q^*\alpha_i^R\big)\Big]
\end{aligned}$$

where the last line is a combination of descendents on both factors of $M \times M$. Since chern class is polynomial in chern characters, we can write down the diagonal class as

$$\Delta_*[M] = \sum_{j\in J} \gamma_j^L \otimes \gamma_j^R$$

where each $\gamma_i^L$ and $\gamma_i^R$ are all tautological classes. This proves that tautological classes generate the cohomology $H^*(M, \mathbb{Q})$.

□

9.3. **Chern class of the tangent bundle.** Since cohomology group is generated by tautological classes, we may express chern classes of the tangent bundle $T_M$ as tautological classes. Recall from deformation theory that

$$T_{M,[F]} \simeq \mathrm{Ext}^1(F, F).$$

Global version of this identification yields

$$T_M \simeq \mathcal{E}xt_\pi^1(\mathcal{F}, \mathcal{F}).$$

Since $\mathcal{H}\mathrm{om}_\pi(F, F) = \mathcal{O}_M$, we have a $K$-theoretic equality

$$T_M = \mathcal{O}_M - \mathrm{R}\mathcal{H}\mathrm{om}_p(\mathcal{F}, \mathcal{F}) \in K^0(M)$$

where we used that $\mathcal{O}_M \simeq \mathcal{H}\mathrm{om}_p(\mathcal{F},\mathcal{F})$. Therefore, we can find chern character of $T_M$ as

$$\begin{aligned}\mathrm{ch}(T_M) &= \mathrm{ch}(\mathcal{O}_M) - \mathrm{ch}\Big(\mathrm{R}\mathcal{H}\mathrm{om}_p(\mathcal{F},\mathcal{F})\Big)\\ &= 1_M - \mathrm{ch}\Big(\mathrm{R}\mathcal{H}\mathrm{om}_p(\mathcal{F},\mathcal{F})\Big)\\ &= 1_M - p_*\Big(\mathrm{ch}^\vee\mathcal{F}\cdot\mathrm{ch}\mathcal{F}\cdot\mathrm{td}(X)\Big)\end{aligned}$$

Consider

$$\begin{aligned}\Delta_*\mathrm{td}(X) &= \Delta_*1_X + (1-g)[\mathrm{pt}]\otimes[\mathrm{pt}]\\ &= 1_X\otimes[\mathrm{pt}] + [\mathrm{pt}]\otimes 1_X + \sum_{i=1}^{g}\Big(f_i\otimes e_i - e_i\otimes f_i\Big) + (1-g)[\mathrm{pt}]\otimes[\mathrm{pt}]\end{aligned}$$

where $\{e_i, f_i\}$ is a symplectic basis[20] of $H^1(X,\mathbb{Q})$. Using the similar trick as before, we have

$$\begin{aligned}\mathrm{ch}(T_M) = 1_M &- \sum_{k_1,k_2\geq 0}(-1)^{k_1}\Big(\mathrm{ch}_{k_1}(1_X)\mathrm{ch}_{k_2}([\mathrm{pt}]) + \mathrm{ch}_{k_1}([\mathrm{pt}])\mathrm{ch}_{k_2}(1_X) + (1-g)\mathrm{ch}_{k_1}([\mathrm{pt}])\mathrm{ch}_{k_2}([\mathrm{pt}])\Big)\\ &- \sum_{k_1,k_2\geq 0}(-1)^{k_1}\sum_{i=1}^{g}\Big(\mathrm{ch}_{k_1}(f_i)\mathrm{ch}_{k_2}(e_i) - \mathrm{ch}_{k_1}(e_i)\mathrm{ch}_{k_2}(f_i)\Big).\end{aligned} \tag{3}$$

This gives very explicit description of chern character of $T_M$ in terms of tautological classes. One can also find chern classes of $T_M$ from this formula though it becomes more complicated.

**Remark 79.** In general, tautological classes depend on the choice of a universal bundle $\mathcal{F}$ even though we have omitted this from the notation for simplicity. However, above formula implies that tautological class on the right hand side of the equality is necessarily independent on the choice of $\mathcal{F}$ because $\mathrm{ch}(T_M)$ is intrinsically defined.

9.4. **Tautological relations.** We would like to introduce some strategy to produce relations among tautological classes. When it comes to relations, chern classes have an advantage over chern characters because chern classes of a vector bundle vanish beyond the rank of a vector bundle. If we can produce some vector bundle over the moduli space whose chern classes we can write down in terms of tautological cases, then we obtain some relations.

Let $M = M(r,d)$ be a moduli space of stable bundles with coprime $(r,d)$ over a curve $X$ of genus $g$. Since moduli space is $r$-periodic with respect to degree $d$, we may assume that $d$ is in a range

$$(2g-1)r < d < (2g-2)r$$

or equivalently

$$2g-1 < \mu < 2g-2.$$

[20]This means that all intersection paring vanishes except $\langle e_i, f_i\rangle = -\langle f_i, e_i\rangle = 1$.

In such a range of slope, for any $F \in M(r,d)$ we have

$$H^1(X,F) = \mathrm{Hom}(F, K_X)^\vee = 0$$

due to stability of $F$ and $K_X$. Since higher cohomology vanishes for all $F \in M$, direct push forward $p_*\mathcal{F}$ is a vector bundle on $M$ of rank $\chi(F) = d + r(1-g)$. This implies that

$$c_k(p_*\mathcal{F}) = c_k(\mathrm{R}p_*\mathcal{F}) = 0, \quad k > d + r(1-g).$$

On the other hand, Grothendieck-Riemann-Roch gives

$$\mathrm{ch}(\mathrm{R}p_*\mathcal{F}) = p_*\Big(\mathrm{ch}\mathcal{F} \cdot q^*\mathrm{td}(X)\Big).$$

Expressing $c_k(p_*\mathcal{F})$ in terms of tautological classes, one obtains tautological relations. Mumford conjecture says that this relation gives entire set of relations when $r = 2$. Precise statement of the conjecture (proven by Kirwan) involves different set of generators and some normalization of the universal bundle $\mathcal{F}$.

9.5. **Moduli of stable bundles of rank 2 with fixed odd determinant.** In this section, we record known results about $M = M_X(2,L)$ where $L$ is a line bundle with an odd degree $d$. Let $\mathcal{F}$ be a universal bundle over $M \times X$. We will see later that $M$ is simply connected hence $H^1(M,\mathbb{Z}) = 0$. Therefore, we may write chern classes of $\mathcal{F}$ as

$$c_1(\mathcal{F}) = 1 \otimes d[\mathrm{pt}] + \phi \otimes 1 \in \Big(H^0(M) \otimes H^2(X)\Big) \oplus \Big(H^2(M) \otimes H^0(X)\Big),$$
$$c_2(\mathcal{F}) = \omega \otimes [\mathrm{pt}] + \psi + \chi \otimes 1 \in \Big(H^2(M) \otimes H^2(X)\Big) \oplus \Big(H^3(M) \otimes H^1(X)\Big) \oplus \Big(H^4(M) \otimes H^0(X)\Big).$$

Diagonal trick also applies to the fixed determinant case. Therefore cohomology ring $H^*(M,\mathbb{Q})$ is generated by classes

$$\omega,\, \phi \in H^2(M), \quad \{\psi_i\}_{1 \le i \le 2g} \in H^3(M), \quad \chi \in H^4(M).$$

where $\psi_i$ are Kunneth components of $\psi \in H^3(M) \otimes H^1(X)$ using the symplectic basis of $H^1(X)$. Note that these classes depends on a choice of a universal bundle $\mathcal{F}$.

We can obtain classes that are independent of $\mathcal{F}$ using the rank 4 endomorphism bundle $\mathrm{End}(\mathcal{F}) := \mathcal{F}^\vee \otimes \mathcal{F}$ whose only non-trivial chern class is

$$\begin{aligned} c_2(\mathcal{F}^\vee \otimes \mathcal{F}) &= 4c_2(\mathcal{F}) - c_1(\mathcal{F})^2 \\ &= 2(2\omega - d\phi) \otimes [\mathrm{pt}] + 4\psi - (\phi^2 - 4\chi) \otimes 1. \end{aligned}$$

Therefore we have cohomology classes

$$\alpha := (2\omega - d\phi) \in H^2(M,\mathbb{Q}), \quad \beta := \phi^2 - 4\chi \in H^4(M,\mathbb{Q})$$

that are independent of choice of $\mathcal{F}$. By squaring the class $\psi$, we also obtain

$$\psi^2 = \gamma \otimes [\mathrm{pt}] \in H^6(M,\mathbb{Q}) \otimes H^2(X).$$

Classes $\alpha,\ \beta,\ \gamma$ are called Nestead classes of $M = M(2,L)$. Nestead classes naturally appears in many intersection theoretic questions. Most importantly, we can express the tangent bundle using $\alpha$ and $\beta$ classes. Recall from deformation theory that

$$T_M = \mathcal{O}_M^{\oplus(1-g)} - \mathrm{R}\mathcal{H}\mathrm{om}_p(\mathcal{F},\mathcal{F}) \in K^0(M).$$

By Grothendieck-Riemann-Roch formula,

$$\mathrm{ch}(T_M) = (1-g)1_M - p_*\Big(\mathrm{ch}(\mathcal{F}^\vee \otimes \mathcal{F}) \cdot q^*\mathrm{td}(X)\Big).$$

On the other hand, we know that

$$c(\mathcal{F}^\vee \otimes \mathcal{F}) = 1 + c_2(\mathcal{F}^\vee \otimes \mathcal{F}).$$

By universal formula between chern class and chern character, we obtain

$$\mathrm{ch}_{2r}(\mathcal{F}^\vee \otimes \mathcal{F}) = \frac{2(-1)^r c_2(\mathcal{F}^\vee \otimes \mathcal{F})^r}{(2r)!} = \frac{2(-1)^r}{(2r)!}\Big(2\alpha \otimes [\mathrm{pt}] + 4\psi - \beta \otimes 1\Big)^r.$$

If we only restrict to even chern character of $T_M$, then $\psi$ class cannot be involved in the formula. One can check easily that

$$\mathrm{ch}_1(T_M) = \alpha, \quad \mathrm{ch}_{2r}(T_M) = \frac{2(g-1)}{(2r)!}\beta^r \text{ for } r > 0.$$

From these formulas and combinatorics, one can also show that

$$\mathrm{td}(M) = e^\alpha \cdot \left(\frac{\sqrt{\beta}/2}{\sinh \sqrt{\beta}/2}\right)^{2g-2}.$$

After we introduce determinant line bundle $\Theta^{\otimes k} \in \mathsf{Pic}(M)$, we can check that

$$c_1(\Theta^{\otimes k}) = k\alpha \in H^2(M,\mathbb{Z}).$$

Then Hirzebruch-Riemann-Roch formula says

$$\chi(M,\Theta^{\otimes k}) = \int_{[M]} e^{(k+1)\alpha} \cdot \left(\frac{\sqrt{\beta}/2}{\sinh \sqrt{\beta}/2}\right)^{2g-2}.$$

In fact, all the intersection number with respect to $\alpha$ and $\beta$ classes are known:

$$\int_{[M]} \alpha^m \beta^n = (-1)^n m! 4^{g-1} b_{g-1-n}, \quad \text{if } m + 2n = 3g-3$$

where

$$\frac{x}{\sin(x)} = \sum_{k\geq 0} b_k x^{2k}, \quad b_i := 0 \text{ for } i < 0.$$

## 10. Determinant line bundle

In this section, we introduce determinant line bundle on the moduli space $M = M(r,d)$ with $\gcd(r,d) = 1$. Definition can also be extended to non-coprime case using descent argument, but we restrict to coprime case for simplicity. See [ABBLT, Section 4] for a more general case and details of the material in this section. Through out the section, we fix a universal bundle $\mathcal{F}$ on $M(r,d) \times X$. Several constructions in this section do depend on the choice of $\mathcal{F}$ while we emphasize that some of the constructions turn out to be independent.

For any vector bundle $V$ on $X$, we define a *determinant line bundle* (or theta line bundle)

$$\mathcal{L}_V := \Big(\det \mathrm{R}p_*\Big(\mathcal{F} \otimes q^*V\Big)\Big)^\vee \in \mathsf{Pic}(M(r,d)).$$

**Proposition 80.** *Let $0 \to V' \to V \to V'' \to 0$ be an exact sequence of vector bundles on $X$. Then we have*

$$\mathcal{L}_V \simeq \mathcal{L}_{V'} \otimes \mathcal{L}_{V''}.$$

*In other words, we have a group homomorphism*

$$K^0(X) \to \mathsf{Pic}(M(r,d)), \quad [V] \mapsto \mathcal{L}_V.$$

*Proof.* Given a short exact sequence $0 \to V' \to V \to V'' \to 0$, we have a distinguished triangle

$$\mathrm{R}p_*\Big(\mathcal{F} \otimes q^*V'\Big) \to \mathrm{R}p_*\Big(\mathcal{F} \otimes q^*V\Big) \to \mathrm{R}p_*\Big(\mathcal{F} \otimes q^*V''\Big) \to \mathrm{R}p_*\Big(\mathcal{F} \otimes q^*V'\Big)[1] \to$$

in $D^b(M(r,d))$. Since determinant is multiplicative under distinguished triangle, we have the desired property. □

**Remark 81.** For a curve $X$, it is known that

$$K^0(X) \xrightarrow{\sim} \mathbb{Z} \times \mathsf{Pic}(X), \quad [V] \mapsto \big(\mathrm{rk}(V), \det(V)\big).$$

Therefore, determinant line bundle $\mathcal{L}_V$ depends on $V$ only through $\mathrm{rk}(V)$ and $\det(V)$.

**Lemma 82.** *Let $[V] \in K^0(X)$ be a $K$-theory class such that $\chi(F \otimes [V]) = 0$ for any $F \in M(r,d)$. Then $\mathcal{L}_V$ is independent on a choice of $\mathcal{F}$.*

*Proof.* Let $N$ be a line bundle on $M(r,d)$ and $\mathcal{F} \otimes p^*N$ be another universal bundle. By computation, we check that

$$\begin{aligned}
\Big(\det \mathrm{R}p_*\Big(\mathcal{F} \otimes p^*N \otimes q^*[V]\Big)\Big)^\vee &= \Big(\det\Big(N \otimes \mathrm{R}p_*\big(\mathcal{F} \otimes q^*[V]\big)\Big)\Big)^\vee \\
&= \Big(N^{\chi(F \otimes [V])} \otimes \det \mathrm{R}p_*\big(\mathcal{F} \otimes q^*[V]\big)\Big)^\vee \\
&= \mathcal{L}_V.
\end{aligned}$$

□

**Remark 83.** Denote by $K^0(X)^\perp$ be a subgroup of $K^0(X)$ defined by $\chi(F \otimes [V]) = 0$ for some $F$. Above lemma says that we have a well-defined homomorphism

$$K^0(X)^\perp \to \mathsf{Pic}(M(r,d))$$

which is independent on a choice of a universal bundle $\mathcal{F}$.

**Lemma 84.** *Let $[V], [W] \in K^0(X)$ be $K$-theory classes of equal rank and degree such that $\chi(F \otimes [V]) = \chi(F \otimes [W]) = 0$. Then we have*

$$\mathcal{L}_V \simeq \mathcal{L}_W \otimes \det{}^*(N)$$

*for some line bundle $N$ on $\mathsf{Pic}^d_X$.*

*Proof.* If $[V]$ and $[W]$ are $K$-theory of equal rank and degree (with possibly different determinant), then the difference is expressed as

$$[V] - [W] = \sum_{i \in I} n_i [\mathcal{O}_{x_i}].$$

for some $n_i \in \mathbb{Z}$ and $x_i \in |X|$ such that $\sum_{i \in I} n_i = 0$. Therefore, it suffices to study $\mathcal{L}_{[\mathcal{O}_x]}$. Let $\mathcal{P}$ be a universal line bundle over $\mathsf{Pic}^d_X \times X$. By definition of the determinant morphism $\det : M(r,d) \to \mathsf{Pic}^d_X$, there is a line bundle $K$ on $M(r,d)$ such that

$$\det \mathcal{F} \simeq (\det \times \mathrm{id}_X)^* \mathcal{P} \otimes p^* K.$$

By computation, we have

$$\begin{aligned} \mathcal{L}_{[\mathcal{O}_x]} &= \Big(\det \mathrm{R}p_* \Big(\mathcal{F} \otimes q^* \mathcal{O}_x\Big)\Big)^\vee \\ &= \Big(\det \mathcal{F}\Big|_{M \times \{x\}}\Big)^\vee \\ &= \det{}^* \mathcal{P}\Big|^\vee_{\mathsf{Pic} \times \{x\}} \otimes K^\vee. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathcal{L}_V \otimes \mathcal{L}_W^\vee &\simeq \bigotimes_{i \in I} \Big(\mathcal{L}_{[\mathcal{O}_{x_i}]}\Big)^{\otimes n_i} \\ &\simeq \det{}^*(N) \otimes K^{\otimes \sum_{i \in I} -n_i} \\ &\simeq \det{}^*(N) \end{aligned}$$

for some line bundle $N$ on $\mathsf{Pic}^d_X$. $\square$

**Remark 85.** Consider a moduli space $M(r,L)$ of stable bundles of rank $r$ with fixed determinant $L$ for some degree $d$ line bundle $L$. By the above property, determinant line bundle construction

$$K^0(X)^\perp \to \mathsf{Pic}(M(r,L))$$

factors through the numerical $K$-group $K^0(X)^{\perp}_{\text{num}}$. Note that numerical $K$-theory class is equivalent to the data of rank and degree. By Riemann-Roch, $(r', d') \in K^0(X)$ is perpendicular to the data $(r, d)$ if and only if

$$rd' + r'd + rr'(1-g) = 0.$$

Since $\gcd(r, d) = 1$, solutions are of the form

$$(r', d') = m \cdot (r, r(g-1) - d), \quad m \in \mathbb{Z}.$$

Combining these two observations, we have a homomorphism

$$\mathbb{Z} \to \mathsf{Pic}(M(r, L)). \tag{4}$$

Recall that this homomorphism is independent of a choice of a universal bundle. Theorem of Drezet-Narasimhan [DN] says that this homomorphism is indeed an isomorphism

**Theorem 86.** *The homomorphism* (4) *is an isomorphism.*

**Definition 87.** We define a *theta line bundle* $\Theta$ as a line bundle corresponding to 1 via the isomorphism (4). For each $k \geq 0$, we call $\Theta^{\otimes k}$ level $k$ theta line bundle. We define a *space of theta functions with rank $r$ and level $k$* (with determinant $L$) as

$$H^0(M(r, L), \Theta^{\otimes k})$$

whose dimension gives a *Verlinde number.* Note that this depends on $L$ only through $\deg(L) = d$. Even though we defined it only for the coprime case $\gcd(r, d) = 1$, definition can be extended to arbitrary cases. In fact, the original work of Verlinde is related to the case of trivial determinant which can never be coprime as long as $r \geq 2$.

Since $M(r, L)$ is a smooth unirational variety of Picard rank 1, it is necessarily a Fano variety. Most basic invariant of a Fano variety is the Fano index which we compute below.

**Proposition 88.** *Moduli space $M(r, L)$ is a smooth Fano variety of index 2, i.e., we have*

$$K_{M(r,L)} \simeq \Theta^{-2}.$$

*Proof.* We need show that

$$c_1(T_{M(r,L)}) = 2 \cdot c_1(\Theta).$$

By Grothendieck-Riemann-Roch,

$$\begin{aligned} c_1(\Theta) &= -p_*\Big[\mathrm{ch}\mathcal{F} \cdot q^*\big(\mathrm{ch}(V)\,\mathrm{td}(X)\big)\Big]_2 \\ &= -p_*\Big[\mathrm{ch}\mathcal{F} \cdot q^*\big(r \cdot 1_X - d[\mathrm{pt}]\big)\Big]_2 \\ &= -r \cdot \mathrm{ch}_2(1_X) + d \cdot \mathrm{ch}_1([\mathrm{pt}]). \end{aligned}$$

For a moment, consider a moduli space $M(r,d)$ rather than $M(r,L)$. By the chern character formula (3) of a tangent bundle of $M(r,d)$, we have

$$\begin{aligned}\mathrm{ch}_1(T_{M(r,d)}) &= 2\Big( -\mathrm{ch}_0([\mathrm{pt}])\mathrm{ch}_2(1_X) + \mathrm{ch}_1(1_X)\mathrm{ch}_1([\mathrm{pt}]) - \sum_{i=1}^{g} \mathrm{ch}_1(e_i)\mathrm{ch}_1(f_i)\Big) \\ &= 2\Big( -r\cdot \mathrm{ch}_2(1_X) + d\cdot \mathrm{ch}_1([\mathrm{pt}])\Big) - 2\sum_{i=1}^{g} \mathrm{ch}_1(e_i)\mathrm{ch}_1(f_i).\end{aligned}$$

Recall that $\det : M(r,d) \to \mathsf{Pic}_X^d$ is a smooth morphism and there is an associated exact sequence

$$0 \to T_{\det} \to T_{M(r,d)} \to \det^* T_{\mathsf{Pic}_X^d} \to 0.$$

Let $i : M(r,L) \hookrightarrow M(r,d)$ be an embedding. Then we have

$$\begin{aligned}\mathrm{ch}_1(T_{M(r,L)}) &= i^*\mathrm{ch}_1(T_{\det}) \\ &= i^*\mathrm{ch}_1(T_{M(r,d)}) - i^*\det^*\mathrm{ch}_1(T_{\mathsf{Pic}_X^d}) \\ &= i^*\mathrm{ch}_1(T_{M(r,d)}).\end{aligned}$$

Third equality follows from the fact that $i\circ\det$ is a constant map.[21] To finish the proof, it suffices to prove that

$$\sum_{i=1}^{g} \mathrm{ch}_1(e_i)\mathrm{ch}_1(f_i)$$

is a cohomology class pulled back from $\mathsf{Pic}_X^d$ hence become trivial after pulling back by $i^*$. This follows because the given tautological class is independent of a choice of a universal bundle $\mathcal{F}$ and depends on $\mathcal{F}$ only through $\det\mathcal{F}$. $\square$

**Corollary 89.** *Verlinde number can be computed as an Euler characteristics, i.e.,*

$$h^0(M(r,L),\Theta^{\otimes k}) = \chi(M(r,L),\Theta^{\otimes k}), \quad k\geq 0.$$

*Proof.* Since $M(r,L)$ is a smooth Fano variety and $\Theta$ is an ample generator, this follows from Kodaira vanishing theorem. $\square$

Below is a key property of the determinant line bundle.

**Proposition 90.** *Let $V$ be a vector bundle such that $\chi(F\otimes[V]) = 0$ for any $F\in M(r,d)$. There is a canonical section $s_V \in H^0(M(r,d),\mathcal{L}_V)$ that vanishes on the locus of $F\in M(r,d)$ with $h^0(X,F\otimes V) = h^1(X,F\otimes V) \neq 0$.*

*Proof.* Since $p : M\times X \to M$ is smooth of dimension 1, we have

$$\mathrm{R}p_*\Big(\mathcal{F}\otimes q^*V\Big) \xrightarrow{\sim} \big[E_0 \xrightarrow{\sigma} E_1\big] \in D^b(M).$$

[21]We can also use the fact that $\mathsf{Pic}_X^d$ has a trivial tangent bundle because it is an abelian variety.

By assumption, we have $\mathrm{rk}(E_0) = \mathrm{rk}(E_1) =: r > 0$. Therefore, we have

$$\mathcal{L}_V \simeq \det(E_1) \otimes \det(E_0)^\vee$$

with a canonical section $s_V \in H^0(M(r,d), \mathcal{L}_V)$ induced from

$$\det(\sigma) : \det(E_0) \to \det(E_1).$$

Note that $s_V$ vanishes (set theoretically) over the locus where $\sigma : E_0 \to E_1$ has rank strictly smaller than $r$. Equivalently, $s_V$ vanishes at $F \in M(r,d)$ where $h^0(X, F \otimes V) = h^1(X, F \otimes V) \neq 0$. $\square$

**Remark 91.** The above theorem holds analogously for the moduli space $M(r,L)$ of fixed determinant. Fix any positive integer $k$. Then for any vector bundle $V$ of $\mathrm{ch}(V) = k \cdot (r, r(g-1) - d)$, we have a section

$$s_V \in H^0(M(r,L), \Theta^{\otimes k}).$$

We used the above notation because $\mathcal{L}_V \simeq \Theta^{\otimes k}$ for any given such $V$. Even though the line bundle $\mathcal{L}_V$ on $M(r,L)$ does not depend on $V$, the section $s_V$ does. This observation can be used to produce various sections for the level $k$ theta line bundle.

Recall that $\Theta$ is an ample line bundle on $M(r,L)$ hence we have a projective embedding

$$|\Theta^{\otimes k}| : M(r,L) \hookrightarrow \mathbb{P}^N$$

for sufficiently large $k >> 0$. In fact it suffices to look at sections of the form $s_V$ to obtain a projective embedding by Thereom of [AK] below.

**Theorem 92.** *For every $k >> 0$, sections of $\Theta^{\otimes k}$ of the form $s_V$ induce a projective embedding.*

## 11. Moduli of Thaddeus pairs

In this section, we introduce the work of Thaddeus [T] on $\sigma$-(semi)stable pairs and their moduli space. We assume that $X$ is a smooth projective curve of genus $g \geq 2$.

### 11.1. $\sigma$-(semi)stable pairs.

Let $X$ be a smooth projective curve. We define a *pair* as a data of a vector space $V$ and a coherent sheaf $F$ together with a morphism $\phi : V \otimes \mathcal{O}_X \to F$. We often denote the pair simply by $(V,F)$. Define a category $\mathrm{Pair}(X)$ of pairs where we define a morphism between pairs as follows:

$$\mathrm{Mor}\Big(V \otimes \mathcal{O}_X \xrightarrow{\phi} F\,,\, V' \otimes \mathcal{O}_X \xrightarrow{\phi'} F'\Big) := \big\{ f : V \to V',\ g : F \to F' \,\big|\, \phi' \circ (f \otimes 1) = g \circ \phi \big\}.$$

One can check that $\mathrm{Pair}(X)$ is naturally an abelian category. For each coherent sheaf, there is a definition of Hilbert polynomial $P(F) = \mathrm{rk}(F)\ell + \chi(F)$. Similarly, we define a *Hilbert polynomial* *of a pair* with respect to a stability parameter $\sigma \in \mathbb{R}_{>0}$ as

$$\begin{aligned} P_\sigma(V,F) &:= P(F) + \dim(V)\cdot\sigma \\ &= \mathrm{rk}(F)m + \big(\chi(F) + \dim(V)\cdot\sigma\big). \end{aligned}$$

Note that $P_\sigma$ satisfies additivity under the short exact sequence and positivity of the leading coefficient for non-trivial pairs. We also define a *slope* of a pair as

$$\mu_\sigma(V,F) := \begin{cases} \frac{\deg(F)+\dim(V)\sigma}{\mathrm{rk}(F)}, & \text{if } \mathrm{rk}(F) > 0, \\ \infty, & \text{if } \mathrm{rk}(F) = 0. \end{cases}$$

We make the following definition which is slightly different and more general than the original definition of Thaddeus [T].

**Definition 93.** We say that a pair $(V,F)$ is *$\sigma$-(semi)stable* if for every non-trivial proper subpair $(V',F')$ we have

$$\mu_\sigma(V',F')\,(\leq)\,\mu_\sigma(V,F).$$

**Remark 94.** We record several basic properties of $\sigma$-(semi)stability of pairs.

(1) A pair $[V\otimes\mathcal{O}_X \xrightarrow{\phi} F]$ is automatically $\sigma$-(semi) if $\mathrm{rk}(F) = 0$.
(2) If a pair $[V\otimes\mathcal{O}_X \xrightarrow{\phi} F]$ is $\sigma$-(semi)stable with $\mathrm{rk}(F) > 0$, then $F$ is a vector bundle and $\phi$ induces an injection $V \hookrightarrow H^0(X,F)$.
(3) A pair $[V\otimes\mathcal{O}_X \xrightarrow{\phi} F]$ is $\sigma$-(semi)stable if and only if $F$ is (semi)stable.

We can prove various properties for pairs that are analogous to (semi)stable sheaves.

**Proposition 95.** *Let $X$ be a smooth projective curve and $\sigma \in \mathbb{R}_{>0}$. We have the following properties for $\sigma$-(semi)stable pairs.*

(1) *For each slope $\mu \in \mathbb{R}\sqcup\infty$, we have an abelian category $\mathrm{Pair}(X)^{\sigma-ss}_\mu$.*
(2) *If $\mu_1 > \mu_2$, then $\mathrm{Hom}\Big(\mathrm{Pair}(X)^{\sigma-ss}_{\mu_1}, \mathrm{Pair}(X)^{\sigma-ss}_{\mu_2}\Big) = 0$.*
(3) *Every $\sigma$-stable pairs is a simple object.*
(4) *For every non-trivial pair, there exists a unique Harder-Narasimhan filtration by subpairs.*
(5) *For every $\sigma$-semistable pair, there exists a Jordan-Holder filtration whose graded pieces are well-defined.*

*Proof.* All statements can be proven analogously as the case of (semi)stable sheaves. $\square$

11.2. **Moduli functor.** Fix a vector space $V$ and a topological type $(r,d)$ with $r \geq 0$. Define a moduli functor

$$\mathrm{Pair}(X)_{V,(r,d)} : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$$

by sending any scheme $S$ of finite type to

$$\left\{V \otimes \mathcal{O}_{S\times X} \xrightarrow{\Phi} \mathcal{F} \,\middle|\, \mathcal{F} : S-\text{flat coherent sheaves of type } (r,d)\right\} \Big/ \sim .$$

Isomorphism between two families of pairs are defined as

$$\begin{tikzcd} V \otimes \mathcal{O}_{S\times X} \arrow[r, "\Phi"] \arrow[d, equal] & \mathcal{F} \arrow[d, "\wr"] \\ V \otimes \mathcal{O}_{S\times X} \arrow[r, "\Phi'"'] & \mathcal{F}'. \end{tikzcd}$$

**Proposition 96.** *Let $S$ be a scheme of finite type and $[V \otimes \mathcal{O}_{S\times X} \xrightarrow{\Phi} \mathcal{F}]$ be a $S$-flat family of pairs. Then*

$$S^{\sigma-ss} := \left\{ s \in |S| \,\middle|\, [V \otimes \mathcal{O}_X \to F_s] \text{ is } \sigma\text{-semistable}\right\}$$

*is a Zariski open subset of $S$. The same is true if we replace $\sigma$-semistability with $\sigma$-stability.*

*Proof.* Note that there are only finitely many numerical data of $\dim(V'') = n''$ and $(r'', d'')$ that can destabilize some pairs parametrized by $S$. Therefore it suffices to show that the locus destabilized by this data is Zariski closed. Consider a Grassmannian $\mathrm{Gr}(n', V)$ with $n' := \dim(V) - n''$ and a relative Quot scheme $\mathsf{Quot}_{S\times X/S}(\mathcal{F}, (r'', d''))$. We consider a projective morphism

$$\mathrm{Gr}(n', V) \times_{\mathrm{Spec}\mathbb{C}} \mathsf{Quot}_{S\times X/S}(\mathcal{F}, (r'', d'')) \to S.$$

Above each $s \in |S|$, the fiber is given by a pair

$$V' \subseteq V, \quad F_s \twoheadrightarrow F''.$$

To obtain a quotient of a pair $[V \otimes \mathcal{O}_X \to F_s]$, we need the composition morphism

$$V' \otimes \mathcal{O}_X \to V \otimes \mathcal{O}_X \to F_s \to F''$$

to be zero. This defines a closed subscheme

$$Z \hookrightarrow \mathrm{Gr}(n', V) \times_{\mathrm{Spec}\mathbb{C}} \mathsf{Quot}_{S\times X/S}(\mathcal{F}, (r'', d'')).$$

It is clear from the construction that the closed subset $\mathrm{image}(Z \to S)$ is the locus that is destabilized by quotient pairs with $\dim(V'') = n''$ and type $(r'', d'')$. $\square$

The above proposition defines an open subfunctor

$$\mathrm{Pair}(X)^{\sigma-ss}_{V,(r,d)} \subseteq \mathrm{Pair}(X)_{V,(r,d)}$$

of $\sigma$-semistable pairs. From now, we assume that $r > 0$.

**Proposition 97.** *Let $D$ be an effective divisor of degree $k$. There is a closed embedding between functors*

$$\mathrm{Pair}(X)^{\sigma-ss}_{V,(r,d)} \hookrightarrow \mathrm{Pair}(X)^{\sigma-ss}_{V,(r,d+rk)}$$

*defined by*

$$[V \otimes \mathcal{O}_X \xrightarrow{\phi} F] \mapsto [V \otimes \mathcal{O}_X \xrightarrow{\phi} F \xrightarrow{s_D} F(D)].$$

*Proof.* We show that the map sends $\sigma$-semistable pairs to again $\sigma$-semistable pairs. For notational simplicity, we denote the above mapping as $(V, F) \mapsto (V, F(D))$. Note that there is a short exact sequence between pairs

$$0 \to (V, F) \to (V, F(D)) \to (0, F(D)|_D) \to 0.$$

Consider any subpair $(V_2, F_2) \subseteq (V, F(D))$. This induces an injection between short exact sequences of pairs

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V_1, F_1) & \longrightarrow & (V_2, F_2) & \longrightarrow & (V_3, F_3) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (V, F) & \longrightarrow & (V, F(D)) & \longrightarrow & (0, F(D)|_D) & \longrightarrow & 0. \end{array}$$

Here $(V_3, F_3)$ is defined as an image of a composition morphism

$$(V_2, F_2) \to (V, F(D)) \to (0, F(D)|_D).$$

Note that $V_3 = 0$ and $V_1 = V_2$. Since $F_3$ is a quotient of $F_2|_D$, it is zero dimensional sheaf of length at most $r_2 k$. We need to check that

$$\frac{d_2 + \sigma n_2}{r_2} = \frac{(d_1 + d_3) + \sigma(n_1 + n_3)}{r_1 + r_3} \leq \frac{d + \sigma n}{r} + k = \mu_\sigma(V, F(D)).$$

This follows from the inequality

$$\frac{d_1 + \sigma n_1}{r_1} \leq \frac{d + \sigma n}{r}$$

due to $\sigma$-semistability of $(V, F)$ together with

$$r_3 = n_3 = 0, \quad d_3 \leq r_2 k.$$

Conversely, a pair $[V \otimes \mathcal{O}_X \xrightarrow{\phi'} F']$ in $\mathrm{Pair}(X)^{\sigma-ss}_{V,(r,d+rk)}$ comes from some $[V \otimes \mathcal{O}_X \xrightarrow{\phi} F]$ in $\mathrm{Pair}(X)^{\sigma-ss}_{V,(r,d)}$ if and only if the composition

$$V \otimes \mathcal{O}_X \xrightarrow{\phi'} F' \to F'|_D$$

is zero which is a closed condition. We leave the details to the reader. □

The above proposition allows us to reduce the construction for the moduli space of $\sigma$-semistable pairs to that of arbitrarily large degree $d$. This reduction is useful due to the next proposition.

**Proposition 98.** *Suppose that $d >> d(g, r)$. If $[V \otimes \mathcal{O}_X \to F]$ is $\sigma$-semistable pair of type $(r, d)$, then $F$ is a globally generated vector bundle with $H^1(X, F) = 0$.*

*Proof.* Suppose for the contradiction that $H^1(X, F) \neq 0$. Then there is a non-trivial morphism $f : F \twoheadrightarrow F'' \subseteq K_X$ where $F''$ is a line bundle of degree $d'' \leq 2g-2$. This induces a quotient between pairs $(V, F) \twoheadrightarrow (0, F'')$. By $\sigma$-semistability of a pair $(V, F)$, we have

$$\frac{d + \dim(V) \cdot \sigma}{r} \leq d'' \leq 2g - 2.$$

This implies that

$$d \leq r(2g-2) - \dim(V) \cdot \sigma \leq r(2g-2).$$

Since $d >> d(g, r)$, this is a contradiction. Similarly, one can show that $H^1(X, F(-x)) = 0$ for any $x \in |X|$ which also proves a global generation of a vector bundle $F$. □

**Proposition 99.** *A functor* $\mathrm{Pair}(X)^{\sigma-ss}_{V,(r,d)}$ *is of finite type.*

*Proof.* We may assume that $d >> d(r, g)$. This implies that every $F$ in a $\sigma$-semistable pair $[V \otimes \mathcal{O}_X \to F]$ is a globally generated bundle of $H^1(X, F) = 0$. By the global generation, all vector bundles $F$ in such pairs are parametrized by a scheme of finite type, for instance an open subset of appropriate Quot scheme. Moreover, for each such $F$, morphisms $V \otimes \mathcal{O}_X \to F$ can be also parametrized by a scheme of finite type. Combining these two, we can construct a scheme of finite type that parametrize all pairs $[V \otimes \mathcal{O}_X \to F]$ where $F$ is a globally generated vector bundles with $H^1(X, F) = 0$. Since $\sigma$-semistability is an open condition, there is a Zariski open subset that parametrizes all $\sigma$-semistable pairs of type $(r, d)$. □

11.3. **Construction of moduli space.** From now, we restrict to the case with $\dim(V) = 1$ and omit $V$ from the notation.

**Theorem 100.** *A functor* $\mathrm{Pair}(X)^{\sigma-ss}_{(r,d)}$ *is corepresented by a projective scheme* $P^{\sigma-ss}_X(r, d)$*. Moreover, a functor* $\mathrm{Pair}(X)^{\sigma-st}_{(r,d)}$ *is represented by a quasi-projective scheme* $P^{\sigma-st}_X(r, d) \subseteq P^{\sigma-ss}_X(r, d)$*.*

**Remark 101.** There is a determinant morphism

$$\det : P^{\sigma-ss}_X(r, d) \to \mathsf{Pic}^d_X.$$

From this, we define $P^{\sigma-ss}_X(r, L)$ as a fiber product.

In later sections, we restrict to the case of $P^{\sigma-ss}_X(2, L)$ where $L$ is a line bundle of odd degree $d$. We devote the rest of this subsection to the construction of moduli space in this special case only.

We may and do assume $d >> d(g, 2)$ so that every $\sigma$-semistable pair $[\mathcal{O}_X \to F]$ of $\mathrm{ch}(F) = (2, d)$ satisfies that $F$ is a globally generated vector bundle with $H^1(X, F) = 0$. Fix a vector space $V$ of dimension $\chi(F) = d + 2(1-g)$. We remark that the notation $V$ is now used for a different object from the previous section. Let $[\mathcal{O}_X \xrightarrow{\phi} F]$ be a $\sigma$-semistable pair of type $(2, L)$ meaning that $\mathrm{rk}(F) = 2$ and $\det(F) = L$. We fix an isomorphism $s : V \xrightarrow{\sim} H^0(X, F)$. For this data of pair together with an isomorphism, we define a purely linear algebraic data

$$(\Lambda^2 s, s^{-1}\phi) \in \mathbb{P}\big(\mathrm{Hom}(\Lambda^2 V, H^0(L))\big) \times \mathbb{P}(V).$$

**Lemma 102.** *Linear algebraic data $(\Lambda^2 s, s^{-1}\phi)$ can recover a pair $[\mathcal{O}_X \xrightarrow{\phi} F]$ and an isomorphism $s : V \xrightarrow{\sim} H^0(X, F)$.*

*Proof.* Since $F$ is a globally generated vector bundle of rank 2, we have a surjection

$$V \otimes \mathcal{O}_X \xrightarrow{s\otimes 1} H^0(X, F) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} F.$$

By universal property of Grassmannian $\mathrm{Gr}(V, 2)$, this is equivalent to the data of a morphism $X \to \mathrm{Gr}(V, 2)$. By Plucker embedding, we have a morphism to a projective space

$$X \to \mathrm{Gr}(\Lambda^2 V, 1).$$

Note that this morphism is determined by $\Lambda^2 s$ and can be used to recover a vector bundle $F$ with a surjection $V \otimes \mathcal{O}_X \twoheadrightarrow F$ inducing an isomorphism $s : V \xrightarrow{\sim} H^0(X, F)$. It is clear that $s^{-1}\phi$ can be used to recover the section $\phi \in H^0(X, F)$. □

Define a $\mathbb{Q}$-ample line bundle

$$\mathcal{L}_\sigma := \mathcal{O}(\chi + \sigma, 2\sigma) \in \mathsf{Pic}\Big(\mathbb{P}\big(\mathrm{Hom}(\Lambda^2 V, H^0(L))\big) \times \mathbb{P}(V)\Big).$$

Next proposition compares $\sigma$-semistability with GIT semistability with respect to $\mathcal{L}_\sigma$.

**Proposition 103.** *Suppose that $[\mathcal{O}_X \to F]$ is $\sigma$-semistable pair of type $(2, L)$ and let $s : V \xrightarrow{\sim} H^0(X, F)$. Then the corresponding linear algebraic data $(\Lambda^2 s, s^{-1}\phi)$ is GIT semistable with respect to a $\mathrm{SL}(V)$-linearized $\mathbb{Q}$-ample line bundle $\mathcal{L}_\sigma$.*

*Proof.* We first characterize a condition for a point

$$(a, b) \in \mathbb{P}\big(\mathrm{Hom}(\Lambda^2 V, H^0(L))\big) \times \mathbb{P}(V)$$

to be GIT semistable with respect to $\mathcal{L}_\sigma$. To apply Hilbert-Mumford numerical criterion, pick any non-trivial 1-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{SL}(V)$. This is equivalent to a non-trivial weight decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V_n \mathbf{t}^n \quad \text{such that} \quad \sum_{n\in\mathbb{Z}} n \cdot \dim(V_n) = 0.$$

We define a filtration of $V$ using $V_{\le n}$ as before. We first find a limit point

$$(\overline{a}, \overline{b}) := \lim_{t\to\infty} \lambda(t) \cdot (a, b).$$

[22] Denote weight decomposition of $b$ as

$$b = \sum_{n\in\mathbb{Z}} b_n \in \bigoplus_{n\in\mathbb{Z}} V_n \mathbf{t}^n.$$

Clearly, $\overline{b} = b_k$ where $k \in \mathbb{Z}$ is the maximal element such that $b_k \neq 0$. Therefore weight of $\mathcal{O}(2\sigma)$ at $\overline{b}$ is equal to $-2\sigma \cdot k$. Similarly, consider a decomposition of $a$ as

$$a = \sum_{i\leq j\in\mathbb{Z}} a_{ij} \in \bigoplus_{i\leq j\in\mathbb{Z}} \mathrm{Hom}(V_i\Lambda V_j, H^0(L))\mathbf{t}^{-i-j}.$$

[23] Then weight of $\mathcal{O}(\chi+\sigma)$ at the limit point $\overline{a}$ is equal to $(\chi+\sigma)(i+j)$ where $i \leq j \in \mathbb{Z}$ is chosen so that $i+j$ is minimal among $a_{ij} \neq 0$. Therefore, a point $(a, b)$ is GIT semistable with respect to $\mathcal{L}_\sigma$ if we have

$$(\chi+\sigma)(i+j) - 2\sigma \cdot k \leq 0$$

for any choice of weight decomposition $V = \bigoplus V_n \mathbf{t}^n$ with $\sum n \cdot \dim(V_n) = 0$.

We now check the above numerical criterion for $(\Lambda^2 s, s^{-1}\phi)$ that comes from $\sigma$-semistable pair $[\mathcal{O}_X \xrightarrow{\phi} F]$ together with $s : V \xrightarrow{\sim} H^0(X, F)$. Suppose for the contradiction that it is not GIT semistable with respect to $\mathcal{L}_\sigma$. Then there exists a weight decomposition $V = \bigoplus V_n \mathbf{t}^n$ such that $\sum n \cdot \dim(V_n) = 0$ with the following property. Let $k$ be a maximal integer such that $(s^{-1}\phi)_k \neq 0$. Then for every $i \leq j \in \mathbb{Z}$ satisfying

$$i + j \leq \frac{2\sigma}{\chi+\sigma} \cdot k$$

we have $a_{ij} = 0$. Let $a$ be the minimal so that $V_a \neq 0$ and pick any non zero vector $v_a \in V_a$. Let $M$ be a line bundle defined as an image of a section $s(v_a) \in H^0(X, F)$. We show that $M$ destabilizes a $\sigma$-semistable pair $[\mathcal{O}_X \xrightarrow{\phi} F]$ hence a contradiction. We divide the cases into two: when $\phi \in H^0(X, F)$ factors through $M$ and otherwise.

We only show the case when $\phi$ factors through $M$ hence defining a subpair $(\mathbb{C}, M) \subseteq (\mathbb{C}, F)$ and leave the other case for the reader. For the contradiction, we need to show that

$$\chi(M) + \sigma > \frac{\chi(F)+\sigma}{2} \quad \text{or equivalently} \quad \chi(M) > \frac{d-\sigma}{2}.$$

For this, it suffices to show that $h^0(X, M) > (d-\sigma)/2$ because this implies $h^1(X, M) = 0$ when $d$ is sufficiently large compared to the genus $g$.[24] Let $b$ be a minimal integer such that

$$\dim(V_{\leq b}) > \frac{\chi-\sigma}{2}.$$

One can check that

$$\frac{\chi-\sigma}{2} \cdot a + \frac{\chi+\sigma}{2} \cdot b \leq \sum_{n\in\mathbb{Z}} n \cdot \dim(V_n) = 0$$

[22] Note that to follow the convention of paper of Thaddeus we take limit at $t \to \infty$. Then we have to change the direction of the inequality for the Hilbert-Mumford criterion.

[23] Here $V_i\Lambda V_j$ means $V_i \otimes V_j$ if $i < j$ and $\Lambda^2 V_i$ if $i = j$.

[24] Use Proposition 51 saying $h^0(X, M) \leq [\deg(M) + 1]_+$.

by drawing the graph of $\dim(V_{\le x})$ and interpreting the above choice of $a$ and $b$ geometrically. By computation, we have

$$a+b \le a - \frac{\chi-\sigma}{\chi+\sigma}a = \frac{2\sigma}{\chi+\sigma}a \le \frac{2\sigma}{\chi+\sigma}k.$$

Let $v \in V_{\le b}$ be any vector. Then above computation shows that $s(v_a)\Lambda s(v) = 0$ hence $s(v)$ gives a section of $M$. This proves that $s(V_{\le b}) \subseteq H^0(X,M)$ which proves that $M$ destabilize the pair. $\square$

There is a converse to this statement.

**Proposition 104.** *Let $[\mathcal{O}_X \xrightarrow{\phi} F]$ be a pair of type $(2,L)$ and let $s : V \to H^0(X,F)$ be a linear map. Let $(\Lambda^2 s, v)$ be a corresponding linear algebraic data where $s(v) = \phi$. If $(\Lambda^2 s, v)$ is GIT semistable with respect to a $\mathrm{SL}(V)$-linearlized $\mathbb{Q}$-ample line bundle $\mathcal{L}_\sigma$, then $[\mathcal{O}_X \xrightarrow{\phi} F]$ is $\sigma$-semistable and $s$ is an isomorphism.*

*Proof.* See paper of Thaddues for the proof. $\square$

Now we proceed to the construction for the moduli space via geometric invariant theory. Consider a locally closed subset

$$U \subset \mathsf{Quot}_X(V \otimes \mathcal{O}_X, (2,d))$$

that parametrizes quotients $[V \otimes \mathcal{O}_X \twoheadrightarrow F]$ with a globally generated vector bundle $F$ of type $(2,L)$ inducing an isomorphism $V \simeq H^0(X,F)$. There is a clear $\mathrm{SL}(V)$-action on $U$ and an equivariant morphism

$$f \times 1 : U \times \mathbb{P}(V) \to \mathbb{P}\big(\mathrm{Hom}(\Lambda^2 V, H^0(L)\big) \times \mathbb{P}(V)$$

defined as before. Recall that $\mathrm{SL}(V)$-linearized $\mathbb{Q}$-ample line bundle $\mathcal{L}_\sigma$ on the right hand side defines a GIT semistable open locus $V'(\sigma)$. We also have open locus $V(\sigma)$ of $\sigma$-semistable pairs. Previous two propositions imply that

$$V(\sigma) = (f \times 1)^{-1}(V'(\sigma)).$$

Furthermore, since linear algebraic data $(\Lambda^2 s, s^{-1}\phi)$ determines a pair together with an isomorphism $s : V \xrightarrow{\sim} H^0(X,F)$, restriction morphism

$$(f \times 1) : V(\sigma) \to V'(\sigma)$$

is injective. In fact, valuative critrion of properness shows that the restriction morphism $(f \times 1)$ is a finite morphism onto image. The following proposition of Gieseker can be used to produce a good quotient of $V(\sigma)$ using the good quotient of $V'(\sigma)$.

**Proposition 105.** *Let $G$ be a reductive group and $f : M_1 \to M_2$ be a finite $G$-equivariant morphism. Suppose that a good quotient $M_2//G$ exists. Then a good quotient $M_1//G$ exists and the induced morphism $M_1//G \to M_2//G$ is finite.*

Therefore, $V(\sigma)$ has a good quotient such that the induced morphism

$$V(\sigma)//\mathrm{SL}(V) \to V'(\sigma)//\mathrm{SL}(V)$$

is finite morphism onto image. Since $V'(\sigma)//\mathrm{SL}(V)$ is projective from GIT construction, so is $V(\sigma)//\mathrm{SL}(V)$ which is clearly a coarse moduli space of $\sigma$-semistable pairs of type $(2, L)$. This constructs a projective coarse moduli space

$$P_X^{\sigma-ss}(2, L) = V(\sigma)//\mathrm{SL}(V).$$

By descent argument, one can also prove that there is a universal pair $[\mathcal{O}_{P^{\sigma-st}\times X} \xrightarrow{\Phi} \mathcal{F}]$ over the stable locus $P_X^{\sigma-st}(2, L) \times X$.

11.4. **Deformation theory of pairs.** Consider a $\sigma$-semistable pair $[\mathcal{O}_X \to F]$. Consider a deformation functor $D_{[\mathcal{O}_X \to F]}$.

**Theorem 106.** *The deformation functor $D_{[\mathcal{O}_X \to F]}$ admits a tangent-obstruction theory with a tangent space* $\mathrm{Hom}(K, F)$ *and an obstruction space* $\mathrm{Ext}^1(K, F)$ *where $K = [\mathcal{O}_X \to F]$ is an object in derived category $D^b(X)$ located in degree* $[0, 1]$.

**Corollary 107.** *Let $P_X^{\sigma-st}(r, d)$ be a moduli space of $\sigma$-stable pairs. Then $P_X^{\sigma-ss}(r, d)$ is smooth of dimension $\chi(\mathcal{O}_X - F, F)$.*

*Proof.* Consider an exact triangle $K \to \mathcal{O}_X \xrightarrow{\phi} F \to K[1]$ in a derived category $D^b(X)$. By taking $\mathrm{Hom}(-, F)$, we get a long exact sequence

$$\begin{aligned}0 \to \mathrm{Ext}^{-1}(K, F) \to \mathrm{Ext}^0(F, F) \to \mathrm{Ext}^0(\mathcal{O}_X, F)&\\ \to \mathrm{Hom}(K, F) \to \mathrm{Ext}^1(F, F) \to \mathrm{Ext}^1(\mathcal{O}_X, F) &\to \mathrm{Ext}^1(K, F) \to 0.\end{aligned}$$

We claim that $\mathrm{Ext}^{-1}(K, F) = \mathrm{Ext}^1(K, F) = 0$, or equivalently $\mathrm{Ext}^0(F, F) \to \mathrm{Ext}^0(\mathcal{O}_X, F)$ is injective and $\mathrm{Ext}^1(F, F) \to \mathrm{Ext}^1(\mathcal{O}_X, F)$ is surjective. For the first claim, suppose that there is a non-zero morphism $f \in \mathrm{Hom}(F, F)$ such that $f \circ \phi = 0$. This induces a morphism between pairs

$$(\mathbb{C}, F) \twoheadrightarrow (0, \mathrm{image}(F)) \subset (\mathbb{C}, F).$$

However, this violates the fact that a -stable pair $(\mathbb{C}, F)$ is simple. The second claim is equivalent to the injectivity of $\mathrm{Hom}(F, K_X) \to \mathrm{Hom}(F, F \otimes K_X)$ which follows from the fact that $\phi$ is an injection between sheaves. This proves that the obstruction space $\mathrm{Ext}^1(K, F) = 0$ vanishes and the tangent space $\mathrm{Hom}(K, F)$ lives in an exact sequence

$$0 \to \mathrm{Ext}^0(F, F) \to \mathrm{Ext}^0(\mathcal{O}_X, F) \to \mathrm{Hom}(K, F) \to \mathrm{Ext}^1(F, F) \to \mathrm{Ext}^1(\mathcal{O}_X, F) \to 0. \tag{5}$$

This implies the dimension formula $\chi(\mathcal{O}_X - F, F)$. $\square$

**Corollary 108.** *Let $P^{\sigma-st}(r,L)$ be a moduli space of $\sigma$-stable pairs with a fixed determinant $L$. Then $P^{\sigma-st}(r,L)$ is smooth of dimension $\chi(\mathcal{O}_X - F, F) - g$.*

*Proof.* The connecting homomorphism $\mathrm{Hom}(K,F) \to \mathrm{Ext}^1(F,F)$ from the previous proof sends deformation of pair $[\mathcal{O}_X \to F]$ to deformation of bare vector bundle $F$. We claim that composition

$$\mathrm{Hom}(K,F) \to \mathrm{Ext}^1(F,F) \xrightarrow{\mathrm{tr}} H^1(\mathcal{O}_X)$$

is surjective. By the splitting

$$\mathrm{Ext}^1(F,F) = \mathrm{Ext}^1(F,F)_0 \oplus H^1(\mathcal{O}_X)$$

and the exact sequence (5), it suffices to show that $\mathrm{Ext}^1(F,F)_0 \to \mathrm{Ext}^1(\mathcal{O}_X,F)$ is surjective. This follows again due to injectivity of sheaf morphism $K_X \otimes F^* \to K_X \otimes \mathrm{End}(F)_0$. Since surjection $\mathrm{Hom}(K,F) \twoheadrightarrow H^1(\mathcal{O}_X)$ can be identified with a differential of a determinant morphism $P_X^{\sigma-st}(r,d) \to \mathsf{Pic}_X^d$, determinant morphism is smooth. This proves the corollary. Furthermore, tangent space of $P_X^{\sigma-st}(r,L)$ lives in an exact sequence

$$0 \to \mathrm{Ext}^0(F,F) \to \mathrm{Ext}^0(\mathcal{O}_X,F) \to T_{P_X^{\sigma-st}(r,L),[\mathcal{O}_X\to F]} \to \mathrm{Ext}^1(F,F)_0 \to \mathrm{Ext}^1(\mathcal{O}_X,F) \to 0.$$

□

**Remark 109.** By Riemann-Roch, $P_X^{\sigma-st}(2,L)$ with $\deg(L) = d$ is smooth of dimension $d+g-2$.

## 12. Wall-crossing phenomenon for pairs

From now and until the end of the note, we assume that $X$ is a smooth projective curve of genus $g \geq 2$. We also fix a rank $r = 2$ and determinant to be a line bundle $L$ of odd degree $d$ which is sufficiently large compared to the genus $g$. For each stability parameter $\sigma \in \mathbb{R}_{>0}$, we have a projective moduli space $P_\sigma(2,L)$. In this section, we study how the moduli space varies under the change of stability parameter $\sigma$.

Recall that a pair $(\mathbb{C},F)$ of type $(2,L)$ is $\sigma$-semistable if and only if

(1) for every $(0,M) \subset (\mathbb{C},F)$, we have $\deg(M) \leq \dfrac{d+\sigma}{2}$
(2) for every $(\mathbb{C},M) \subset (\mathbb{C},F)$, we have $\deg(M) \leq \dfrac{d-\sigma}{2}$

where $M$ is a line bundle. For numerical reasons due to integrality of $d$ and $\deg(M)$, the notion of $\sigma$-semistability and $\sigma$-stability are the same and stay unchanged in open intervals

$$\sigma \in \Big(d-2(i+1), d-2i\Big), \quad i \in \left[0, \frac{d-1}{2}\right] \cap \mathbb{Z}.$$

However when $\sigma$ crosses odd integers $d-2i$, the notion of $\sigma$-semistability may change and so is the moduli space. This is called wall-crossing phenomenon.

Let $w := \frac{d-1}{2} \in \mathbb{Z}$. When $i = w$, what we meant from the above is $\sigma \in (0,1)$ rather than $\sigma \in (-1,1)$ since $\sigma$ is always positive. Define $P_i$ as a moduli space $P_\sigma(2,L)$ where $\sigma \in (d-2(i+1), d-2i)$. Since $\sigma$-semistability and $\sigma$-stability agrees for such $\sigma$, $P_i$ is smooth projective of dimension $d+g-2$ and is equipped with a universal pair $[\mathcal{O}_{P_i \times X} \xrightarrow{\Phi_i} \mathcal{F}_i]$. We would like to study relations between various moduli spaces

$$P_0\ ,\ P_1\ ,\ \ldots\ , P_{w-1}\ ,\ P_w.$$

12.1. **Extreme cases.** We begin by studying extreme cases, i.e., $P_0$ and $P_w$. We start with $P_w$. Recall that $P_w$ is a moduli space of $\sigma$-semistable pairs where $\sigma = \epsilon$ for sufficiently small positive number $\epsilon$. Therefore, a pair $(\mathbb{C}, F)$ is in $P_w$ if and only if the following conditions are satisfied:

(1) for every $(0,M) \subset (\mathbb{C}, F)$, we have $\deg(M) \leq \frac{d+\epsilon}{2}$ or equivalently $\deg(M) \leq \frac{d}{2}$,
(2) for every $(\mathbb{C}, M) \subset (\mathbb{C}, F)$, we have $\deg(M) \leq \frac{d-\epsilon}{2}$ or equivalently $\deg(M) < \frac{d}{2}$.

The first condition is equivalent to (semi)stability of $F$. On the other hand, the second condition is vacuously true as long as $\phi \in H^0(X,F)$ is non-zero and the first condition is satisfied. Therefore, a pair $[\mathcal{O}_X \xrightarrow{\phi} F] \in P_w$ if and only if $F$ is a stable bundle of type $(2,L)$ and $\phi \neq 0$. Note that $H^1(X,F) = 0$ since we assumed that $d$ is sufficiently large compared to the genus $g$. Therefore, we have a forgetful morphism

$$\pi : P_w \to M(2,L), \quad [\mathcal{O}_X \xrightarrow{\phi} F] \mapsto F$$

where each fiber is canonically isomorphic to a projective space $\mathbb{P}(H^0(X,F))$. Geometry of $P_w$ and $M(2,L)$ are very closely related. If $\mathbb{G}$ is a choice of a universal bundle over $M(2,L) \times X$, then we can identify $P_w$ as a projectivization of a vector bundle $\mathbb{P}(p_*\mathbb{G})$ over $M(2,L)$. Universal pair is then given by $[\mathcal{O}_{P_w \times X} \to (\pi \times 1)^*\mathbb{G}(1)]$ where $\mathcal{O}_{\mathbb{P}}(1)$ is a tautological bundle. This can be used as a bridge between pair moduli space $P_w$ and the usual moduli space $M(2,L)$ of stable bundles which is of our main interest.

Now we consider the other extreme case. We first show that $P_\sigma(2,d) = \emptyset$ if $\sigma > d$ which one might interpret as $P_{-1} = \emptyset$. Suppose for a contradiction that there is a pair $[\mathcal{O}_X \xrightarrow{\phi} F]$ which is $\sigma$-semistable with $\sigma > d$. Let $M := \mathrm{image}(\phi)$ be the image line bundle which is necessarily isomorphic to $\mathcal{O}_X$. By $\sigma$-semistability and a subpair $(\mathbb{C}, M) \subset (\mathbb{C}, F)$, we have $0 = \deg(M) \leq \frac{d-\sigma}{2} < 0$ which is a contradiction. Therefore $P_{-1} = \emptyset$.

From the above observation, $P_0$ is the first potentially non-empty pair moduli space. We do show that $P_0$ is non-empty and in fact just a projective space of dimension $d+g-2$. Suppose that $[\mathcal{O}_X \xrightarrow{\phi} F]$ is a pair in $P_0$ which means that

(1) for every $(0,M) \subset (\mathbb{C}, F)$, we have $\deg(M) \leq \frac{d+(d-\epsilon)}{2}$ or equivalently $\deg(M) < d$,
(2) for every $(\mathbb{C}, M) \subset (\mathbb{C}, F)$, we have $\deg(M) \leq \frac{d-(d-\epsilon)}{2}$ or equivalently $\deg(M) \leq 0$.

Consider an exact sequence between pairs

$$0 \to (\mathbb{C}, \mathrm{image}(\phi)) \to (\mathbb{C}, F) \to (0, \mathrm{coker}(\phi)) \to 0.$$

As before, we know that $\mathrm{image}(\phi) \simeq \mathcal{O}_X$. We can show that $\mathrm{coker}(\phi)$ is torsion free sheaf because otherwise we can construct a subpair $(\mathbb{C}, M) \subset (\mathbb{C}, F)$ with $\deg(M) > 0$. Therefore, $\mathrm{coker}(\phi)$ is necessarily a line bundle which is $L$ because $\deg(F) \simeq L$. This gives a diagram

$$0 \to \mathcal{O}_X \xrightarrow{\phi} F \to L \to 0$$

which is a non-splitting (due to stability again) exact sequence. We leave for the reader to check that whenever we are given such non-splitting extension we get a pair in $P_0$. This proves that

$$P_0 \simeq \mathbb{P}(\mathrm{Ext}^1(L, \mathcal{O}_X)).$$

We can compute the dimension of the Ext group by Riemann-Roch formula because $\mathrm{Hom}(L, \mathcal{O}_X) = 0$ since $d >> d(g)$.

12.2. **Explicit description of wall-crossing diagram.** In the previous section, we have identified two extreme cases where $P_w$ was closely related to the moduli space $M(2, L)$ of stable bundles while the other case $P_0$ was simply a projective space. Wall-crossing then interpolates these two spaces via various birational transformation. In this section, we describe this birational transformation explicitly as a sequence of blow ups and blow downs. In the end, we prove that there a diagram

$$\begin{tikzcd}
 & \widetilde{P}_2 \arrow[dl] \arrow[dr] & & \widetilde{P}_3 \arrow[dl] \arrow[dr] & & \widetilde{P}_4 \arrow[dl] \arrow[dr] & & \widetilde{P}_w \arrow[dl] \arrow[dr] & \\
P_1 \arrow[d] & & P_2 & & P_3 & & \cdots & & P_w \arrow[d] \\
P_0 & & & & & & & & M(2, L)
\end{tikzcd}$$

where all the morphisms are blow up except the last morphism $\pi : P_w \to M(2, L)$ which is a projective bundle. Furthermore, $P_1, \ldots, P_w$ are isomorphic away from codimension at least two so that their Picard groups are naturally identified. This in particular proves that Picard group of $M(2, L)$ is $\mathbb{Z}$ as we used before.

We should note that it can be very difficult to understand each moduli space $P_i$ precisely. In wall-crossing set up, we usually try to study the *difference* between adjacent moduli space $P_{i-1}$ and $P_i$ which are usually more manageable. Let $\sigma_i$ be a stability parameter $\sigma$ that corresponds to a moduli space $P_i$. For $i = 1, \ldots, w$, we have

$$\sigma_i = d - 2i - \epsilon, \quad \sigma_{i-1} = d - 2i + \epsilon$$

for sufficiently small $\epsilon > 0$. There is an open locus

$$U_i := \text{“}P_i \cap P_{i-1}\text{”}$$

which parametrizes pairs which are both $\sigma_i$-stable and $\sigma_{i-1}$-stable. Our goal is to describe the closed locus $P_i \backslash U_i$ and $P_{i-1} \backslash U_i$. For this, we need to introduce some notations. Let $\mathrm{Sym}^i(X)$ be the $i$'th symmetric product of a curve that parametrizes effective divisors $D \subset X$ of length $i$. There is a universal ideal sequence

$$0 \to \mathcal{O}(-\mathcal{D}) \to \mathcal{O}_{\mathrm{Sym}^i(X)\times X} \to \mathcal{O}_{\mathcal{D}} \to 0.$$

We define two vector bundles

$$W_i^- := \mathcal{E}\mathrm{xt}^1_p\Big(\mathcal{O}_{\mathcal{D}}(\mathcal{D}), q^*L(-\mathcal{D})\Big), \quad W_i^+ := \mathcal{E}\mathrm{xt}^1_p\Big(q^*L(-\mathcal{D}), \mathcal{O}(\mathcal{D})\Big)$$

on $\mathrm{Sym}^i(X)$ of rank $i$ and $d+g-1-2i$, respectively. Note that we have an isomorphism $W_i^- \simeq p_*\big(\mathcal{O}_{\mathcal{D}}(-\mathcal{D}) \otimes q^*L\big)$. The next proposition describes exactly how $P_i \backslash U_i$ and $P_{i-1} \backslash U_i$ looks like.

**Proposition 110.**

(1) *There is a family of pairs over* $\mathbb{P}(W_i^+)$ *that describes exactly those pairs in* $P_i \backslash U_i$.
(2) *There is a family of pairs over* $\mathbb{P}(W_i^-)$ *that describes exactly those pairs in* $P_{i-1} \backslash U_i$.

*Proof.* For the first part, we need to describes pairs $(\mathbb{C}, F)$ which are $(d-2i-\epsilon)$-stable but not $(d-2i+\epsilon)$-stable. This is equivalent to having an exact sequence

$$0 \to \mathcal{O}(D) \to F \to L(-D) \to 0$$

where section $\phi \in H^0(X, F)$ factors through $\mathcal{O}(D)$ that cuts out exactly divisor $D$ of length $i$. In other words, such pairs are in one to one correspondence with a choice of $D \in \mathrm{Sym}^i(X)$ and element in a projective space $\mathbb{P}\big(\mathrm{Ext}^1(L(-D), \mathcal{O}(D)\big)$. Doing this construction relatively, we can construct a family of pairs over $\mathbb{P}(W_i^+)$ of the form

$$0 \to \mathcal{O}(\mathcal{D}) \to \mathcal{F} \to L(-\mathcal{D}) \otimes \mathcal{O}_{\mathbb{P}(W_i^+)}(-1) \to 0$$

where have omitted various pull backs from the notation.

For the second part, we need describes pairs $(\mathbb{C}, F)$ which are $(d-2i+\epsilon)$-stable but not $(d-2i-\epsilon)$-stable. This is equivalent to having an exact sequence

$$0 \to M \to F \to L \otimes M^{-1} \to 0$$

with $\deg(M) = d - i$ and $\phi \notin H^0(X, M)$. This implies that $\phi \in H^0(X, F)$ induces a nonzero section for $L \otimes M^{-1}$ that cuts out a divisor $D$ of length $i$. Therefore we may rewrite this sequence as

$$0 \to L(-D) \to F \to \mathcal{O}(D) \to 0.$$

Since a morphism $\mathcal{O}_X \to \mathcal{O}(D)$ vanishes over $D$, section $\phi \in H^0(X, F)$ induces an element in $H^0(X, \mathcal{O}_D(-D) \otimes L)$ which is well-defined up to $\mathbb{C}^*$. On the other hand, we have

$$\begin{aligned} H^0(X, \mathcal{O}_D(-D) \otimes L) &= H^0(D, \mathcal{O}_D(D) \otimes \mathcal{O}_D(-2D) \otimes L_D) \\ &= H^0(D, K_D \otimes (K_X)^\vee|_D \otimes (\mathcal{O}(-2D) \otimes L)_D) \\ &= H^0(D, (K_X)|_D \otimes (\mathcal{O}(2D) \otimes L^\vee)_D)^\vee \\ &= H^0(X, K_X \otimes \mathcal{O}_D(2D) \otimes L^\vee)^\vee \\ &= \mathrm{Hom}_X(L(-D), \mathcal{O}_D(D) \otimes K_X)^\vee \\ &= \mathrm{Ext}^1_X(\mathcal{O}_D(D), L(-D)). \end{aligned}$$

This shows that a pair $(\mathbb{C}, F)$ that is $(d-2i+\epsilon)$-stable but not $(d-2i-\epsilon)$-stable induces a divisor $D \in \mathrm{Sym}^i(X)$ together with an element in $\mathrm{Ext}^1(\mathcal{O}_D(D), L(-D))$ up to $\mathbb{C}^*$. Conversely, if we have $D \in \mathrm{Sym}^i(X)$ together with an extension class

$$0 \to L(-D) \to E \to \mathcal{O}_D(D) \to 0,$$

then we can recover such a pair $(\mathbb{C}, F)$ via the diagram

$$\begin{tikzcd} 0 \arrow[r] & L(-D) \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d, two heads] & \mathcal{O}(D) \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & L(-D) \arrow[r] & E \arrow[r] & \mathcal{O}_D(D) \arrow[r] & 0 \end{tikzcd}$$

Note that $F$ fits into a diagram

$$0 \to F \to E \otimes \mathcal{O}(D) \to \mathcal{O}_D(D) \to 0.$$

We then recover a section $\phi \in H^0(X, F)$ as

$$0 \oplus s_D \in H^0(X, E \oplus \mathcal{O}(D))$$

up to $\mathbb{C}^*$. Doing this construction relatively, we can construct a family of pairs over $\mathbb{P}(W_i^-)$ of the form

$$0 \to L(-\mathcal{D}) \otimes \mathcal{O}_{\mathbb{P}(W_i^-)}(1) \to \mathcal{F} \to \mathcal{O}(\mathcal{D}) \to 0.$$

We refer the details of the proof to the original paper. $\square$

From the above construction, we obtain morphisms

$$\mathbb{P}(W_i^+) \to P_i, \quad \mathbb{P}(W_i^-) \to P_{i-1}$$

which are injective on $\mathbb{C}$-points. One can identify the differential of the morphism explicitly.

**Proposition 111.**

(1) *The morphism $\mathbb{P}(W_i^+) \to P_i$ is a closed embedding with normal bundle $W_i^-(-1)$.*

(2) *The morphism* $\mathbb{P}(W_i^-) \to P_{i-1}$ *is a closed embedding with normal bundle* $W_i^+(-1)$.

*Proof.* See the paper for the proof. $\square$

**Remark 112.** Dimension computation gives

$$\operatorname{codim}(P_i, \mathbb{P}(W_i^+)) = i, \quad \operatorname{codim}(P_{i-1}, \mathbb{P}(W_i^-)) = d - 2i + g - 1.$$

Starting from $P_0 \simeq \mathbb{P}^{d+g-2}$, we can show by induction that $P_0, \dots, P_w$ are all smooth projective rational varieties of dimension $d + g - 2$. Furthermore, $P_1, \dots, P_w$ are isomorphic away from codimension at least 2 which in particular implies that Picard groups are identified for these spaces.

It is easy to check that all pairs in $P_i$ or $P_{i-1}$ are $(d-2i)$-semistable hence inducing morphisms $\pi_i^\pm$ in the following diagram

$$\begin{tikzcd} \mathbb{P}(W_i^+) \arrow[r, hook] & P_i \arrow[dr, "\pi_i^+"'] \arrow[rr, dashed, leftrightarrow] & & P_{i-1} \arrow[dl, "\pi_i^-"] & \mathbb{P}(W_i^-) \arrow[l, hook'] \\ & & P_X^{(d-2i)-ss}(2, L). & & \end{tikzcd}$$

Recall that $\pi_i^\pm$ maps pairs to $S$-equivalence classes with respect to a stability parameter $\sigma = d-2i$. There is a subscheme

$$\operatorname{Sym}^i(X) \hookrightarrow P_X^{(d-2i)-ss}(2, L)$$

that represents $S$-equivalent classes of pairs of the form

$$[\mathcal{O}_X \xrightarrow{\phi \oplus 0} \mathcal{O}(D) \oplus L(-D)].$$

Therefore, $\pi_i^\pm$ maps $\mathbb{P}(W_i^\pm)$ onto $\operatorname{Sym}^i(X)$ and induces an isomorphism away from these locus. One can show from the results collected in the next section that the above diagram is what is called flip in minimal model program when $1 < i \le w$.[25]

Since moduli space $P_X^{(d-2i)-ss}(2, L)$ on the wall does not admit a universal object, it is desirable to compare $P_i$ and $P_{i-1}$ via spaces together with a family of pairs. This can be achieved by considering a blow ups. Consider the blow up diagrams

$$\begin{tikzcd} E_i^+ \arrow[r, hook] \arrow[d] & \widetilde{P}_i^+ \arrow[d] \\ \mathbb{P}(W_i^+) \arrow[r, hook] & P_i, \end{tikzcd} \qquad \begin{tikzcd} \widetilde{P}_{i-1}^- \arrow[d] & E_i^- \arrow[l, hook'] \arrow[d] \\ P_{i-1} & \mathbb{P}(W_i^-). \arrow[l, hook'] \end{tikzcd}$$

From the normal bundle computation for $\mathbb{P}(W_i^\pm)$, we have

$$E_i^- = E_i^+ = \mathbb{P}(W_i^-) \times_{\operatorname{Sym}^i(X)} \mathbb{P}(W_i^+).$$

[25]It depends on $i$ whether $P_i$ flips into $P_{i-1}$ or vice versa.

We will show that two blow ups are isomorphic to each other while matching two exceptional divisors as above. The following proposition is the first step towards this.

**Proposition 113.** *There is a morphism $\widetilde{P}_i^+ \to P_{i-1}$ such that it maps $E_i^+$ to $\mathbb{P}(W_i^-)$ as a projection and it is isomorphism away from these locus.*

*Proof.* Let $[\mathcal{O}_{\widetilde{P}_i^+ \times X} \xrightarrow{\Phi} \mathcal{F}]$ be the pullback of the universal family over $P_i$. We define a desired morphism by modifying this family over $E_i^+$ so that it becomes $\sigma_{i-1}$-stable. By Proposition 110, restriction of the pullback family to $E_i^+ \simeq \mathbb{P}(W_i^-) \times_{\mathrm{Sym}^i(X)} \mathbb{P}(W_i^+)$ is obtained by the short exact sequence

$$0 \to \mathcal{O}_{E_i^+ \times X}(\mathcal{D}) \to \mathcal{F}\Big|_{E_i^+ \times X} \to L(-\mathcal{D}) \otimes \mathcal{O}_{E_i^+}(0,-1) \to 0.$$

Using the above surjection, we define a new family

$$\mathcal{F}' := \ker\Big(\mathcal{F} \twoheadrightarrow \mathcal{F}\Big|_{E_i^+ \times X} \twoheadrightarrow L(-\mathcal{D}) \otimes \mathcal{O}_{E_i^+}(0,-1)\Big).$$

The sheaf $\mathcal{F}'$ over $\widetilde{P}_i^+ \times X$ constructed above is a vector bundle since $L(-\mathcal{D}) \otimes \mathcal{O}_{E_i^+}(0,-1)$ is of homological dimension 1. Furthermore, section $\Phi \in H^0(\widetilde{P}_i^+ \times X, \mathcal{F})$ factors through a section $\Phi' \in H^0(\widetilde{P}_i^+ \times X, \mathcal{F}')$ by construction.

We claim that the modified family of pairs

$$[\mathcal{O}_{\widetilde{P}_i^+ \times X} \xrightarrow{\Phi'} \mathcal{F}]$$

are $\sigma_{i-1}$-stable. Since the family is modified only over $E_i^+$, it is clearly $\sigma_{i-1}$-stable away from $E_i^+$. We study how the modified family looks like over $E_i^+ \times X$. To do so, we apply $- \otimes \mathcal{O}_{E_i^+ \times X}$ to a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to L(-\mathcal{D}) \otimes \mathcal{O}_{E_i^+}(0,-1) \to 0.$$

This yields a long exact sequence

$$0 \to \mathrm{Tor}^1\Big(L(-\mathcal{D}) \otimes \mathcal{O}_{E_i^+}(0,-1), \mathcal{O}_{E_i^+ \times X}\Big) \to \mathcal{F}'\Big|_{E_i^+ \times X} \to \mathcal{F}\Big|_{E_i^+ \times X} \to L(-\mathcal{D}) \otimes \mathcal{O}_{E_i^+}(0,-1) \to 0.$$

On the other hand, we understand the kernel of the last surjection hence obtaining a short exact sequence

$$0 \to \mathrm{Tor}^1_{\widetilde{P}_i^+ \times X}\Big(L(-\mathcal{D}) \otimes \mathcal{O}_{E_i^+}(0,-1), \mathcal{O}_{E_i^+ \times X}\Big) \to \mathcal{F}'\Big|_{E_i^+ \times X} \to \mathcal{O}_{E_i^+ \times X}(\mathcal{D}) \to 0.$$

Furthermore, by the explicit resolution

$$0 \to \mathcal{O}_{\widetilde{P}_i^+ \times X}(-E_i^+) \to \mathcal{O}_{\widetilde{P}_i^+ \times X} \to \mathcal{O}_{E_i^+ \times X} \to 0$$

we can simplify the left most term to obtain

$$0 \to L(-\mathcal{D}) \otimes \mathcal{O}_{E_i^+}(1,0) \to \mathcal{F}'\Big|_{E_i^+ \times X} \to \mathcal{O}_{E_i^+ \times X}(\mathcal{D}) \to 0.$$

This looks exactly like the family obtained in the second part of Proposition 110. After further analysis on the modified family, one can show that this is indeed a pullback of the family from second part of Proposition 110. □

Analogously, we can also do the above procedure on the other side.

**Proposition 114.** *There is a morphism $\widetilde{P}_{i-1}^{-} \to P_i$ such that it maps $E_i^-$ to $\mathbb{P}(W_i^+)$ as a projection and it is isomorphism away from these locus.*

**Proposition 115.** *There is a natural isomorphism between $\widetilde{P}_i^+$ and $\widetilde{P}_{i-1}^-$ such that it identifies $E_i^+$ and $E_i^-$ and the open complement $U_i$.*

*Proof.* By the previous propositions, we have morphisms

$$\widetilde{P}_i^+ \to P_i \times P_{i-1} \leftarrow \widetilde{P}_{i-1}^-.$$

One can show that these morphisms are closed embedding with the same image. Furthermore the image is precisely the closure of the graph of the isomorphism

$$P_i \backslash \mathbb{P}(W_i^+) \simeq P_{i-1} \backslash \mathbb{P}(W_i^-).$$

□

**Remark 116.** Thanks to the above proposition, we may denote the identified spaces as $\widetilde{P}_i$ with the exceptional locus as $E_i$.

**Remark 117.** When $i = 1$, the above propositions imply that there is a morphism

$$P_1 \to P_0$$

obtained as a blow up of the subscheme $X \simeq \mathbb{P}(W_1^-) \hookrightarrow P_0$.

## 13. Proof of Verlinde formula for rank 2

In this section, we study line bundles on moduli spaces and their space of global sections. Verlinde space becomes an example of such a space of global sections. By analyzing ample cone of each moduli space $P_1, \dots, P_w$, we find exactly one $P_i$ that computes the Verlinde number as a Riemann-Roch number. Then Verlinde formula follows from the wall-crossing formula of Riemann-Roch numbers.

13.1. **Line bundles on moduli spaces.** Recall from the last section that we have a blow up diagram

$$\begin{tikzcd} E_1 \arrow[r, hook] \arrow[d] & P_1 \arrow[d] \\ X \simeq \mathbb{P}(W_1^-) \arrow[r, hook] & P_0. \end{tikzcd}$$

Since $P_0 \simeq \mathbb{P}^{d+g-2}$, Picard group of $P_1$ is freely generated by the pull back of the hyperplance section $H$ in $P_0$ and the exceptional divisor $E_1$. We make the following identification

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\sim} \mathsf{Pic}(P_1), \quad (m, n) \mapsto \mathcal{O}_1(m, n) := \mathcal{O}((m+n)H - nE_1).$$

On the other hand, moduli spaces $P_1, \dots, P_w$ are all smooth projective variety which are isomorphic away from codimension at least 2 locus. This allows us to identify

$$\mathsf{Pic}(P_1) \simeq \mathsf{Pic}(P_i), \quad i = 1, \dots, w$$

hence defining $\mathcal{O}_i(m, n) \in \mathsf{Pic}(P_i)$. Also we can identify the space of global sections to define

$$V_{m,n} := H^0(P_1, \mathcal{O}_1(m, n)) \simeq H^0(P_i, \mathcal{O}_i(m, n)), \quad i = 1, \dots, w, \ m, n \in \mathbb{Z}.$$

We define some moduli theoretic line bundles on each $P_i$. Let $[\mathcal{O}_{P_i \times X} \xrightarrow{\Phi_i} \mathcal{F}_i]$ be a universal pair over $P_i \times X$. Pick any point $x \in |X|$. We define two line bundles

$$\det\Big(Rp_*\mathcal{F}_i\Big), \quad \det\Big(\mathcal{F}_i\Big|_{P_i \times \{x\}}\Big) \in \mathsf{Pic}(P_i).$$

Note that the second line bundle is independent on the choice of $x \in |X|$ because Picard group is discrete. These line bundles can be explicitly computed under the identification $\mathsf{Pic}(P_i) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

**Proposition 118.** *For every $i = 1, \dots, w$, we have*

$$\det\Big(Rp_*\mathcal{F}_i\Big) \simeq \mathcal{O}_i(-1, g-d), \quad \det\Big(\mathcal{F}_i\Big|_{P_i \times \{x\}}\Big) \simeq \mathcal{O}_i(0, -1).$$

*Proof.* Since $P_1, \dots, P_w$ are isomorphic away from codimension at least 2, it suffices to prove the statement for $i = 1$ case. Recall from the identification of $P_0 \simeq \mathbb{P}(\mathrm{Ext}^1(L, \mathcal{O}_X))$ that is is equipped with the universal extension

$$0 \to \mathcal{O}_{P_0 \times X} \xrightarrow{\Phi_0} \mathcal{F}_0 \to L(-H) \to 0$$

where $H$ is the hyperplane section of $P_0$. By the proof of Proposition 114 which we have skipped, we can construct a universal family over $P_1 \times X$ by pulling back $[\mathcal{O}_{P_0\times X} \xrightarrow{\Phi} \mathcal{F}_0]$ to $P_1 \times X$ and modify it over the exceptional locus $E_1 \times X$. More precisely, we have a short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_0(E_1) \to \mathcal{O}_{E_1\times X}(\mathcal{D})(-1) \to 0$$

over $P_1 \times X$. Here $\mathcal{O}_{E_1\times X}(-1)$ on the rightmost term refers to a tautological line bundle with respect to $E_1 = \mathbb{P}(W_1^+)$. Using the short exact sequences, we obtain

$$\begin{aligned}\det\left(\mathcal{F}_1\Big|_{P_1\times\{x\}}\right) &\simeq \det\left(\mathcal{F}_0\Big|_{P_0\times\{x\}}(E_1)\right)\otimes\mathcal{O}_{P_1}(-E_1)\\ &\simeq \det\left(\mathcal{F}_0\Big|_{P_0\times\{x\}}\right)\otimes\mathcal{O}_{P_1}(E_1)\\ &\simeq \mathcal{O}_{P_1}(-H+E_1)\\ &\simeq \mathcal{O}_1(0,-1).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\det\left(Rp_*\mathcal{F}_1\right) &\simeq \det\left(Rp_*\big(\mathcal{F}_0(E_1)\big)\right)\otimes\det\left(Rp_*\big(\mathcal{O}_{E_1\times X}(\mathcal{D})(-1)\big)\right)^\vee\\ &\simeq \det\left(Rp_*\mathcal{O}(E_1)\right)\otimes\det\left(Rp_*\big(L\otimes\mathcal{O}(-H+E_1)\big)\right)\otimes\mathcal{O}((g-2)E_1)\\ &\simeq \mathcal{O}((1-g)E_1)\otimes\mathcal{O}((d+1-g)(-H+E_1))\otimes\mathcal{O}((g-2)E_1)\\ &\simeq \mathcal{O}_1(-1,g-d).\end{aligned}$$

□

The proposition below computes the canonical bundle of each moduli space $P_i$ in terms of the identification $\mathsf{Pic}(P_i) \simeq \mathbb{Z}\oplus\mathbb{Z}$.

**Proposition 119.** *For every $i = 1,\ldots,w$, we have*

$$K_{P_i} \simeq \mathcal{O}_i(-3, 4-d-g).$$

*Proof.* It suffices to prove it for $P_1$. By the canonical bundle formula for the blow up, we have

$$\begin{aligned}K_{P_1} &\simeq \mathcal{O}(-(d+g-1)H)\otimes\mathcal{O}((d+g-4)E_1)\\ &\simeq \mathcal{O}_1(-3,4-d-g).\end{aligned}$$

□

Recall that we have a forgetful morphism

$$\pi : P_w \to M(2,L), \quad [\mathcal{O}_X \xrightarrow{\phi} F] \mapsto F.$$

The following proposition computes the pullback of the theta line bundle $\pi^*\Theta$ in terms of the identification $\mathsf{Pic}(P_w) \simeq \mathbb{Z}\oplus\mathbb{Z}$.

**Proposition 120.** *On the moduli space $P_w$, we have*

$$\pi^*\Theta \simeq \mathcal{O}_w(2, d-2).$$

*Proof.* Let $\mathbb{G}$ be a choice of a universal bundle over $M(2,L) \times X$. Then we may identify the forgetful morphism $\pi$ as a projective bundle

$$\pi : P_w \simeq \mathbb{P}(p_*\mathbb{G}) \to M(2,L).$$

Using this identification, the universal pair over $P_w \times X$ can be written as

$$\mathcal{O}_{P_w \times X} \to \mathcal{F} = \mathbb{G}(1)$$

where we omitted pullback $(\pi \times \mathrm{id}_X)^*$ from the notation on the right hand side. By definition of theta line bundle, we have

$$\begin{aligned}\pi^*\Theta &\simeq \det\Big(Rp_*\big(\mathbb{G} \otimes [V]\big)\Big)^\vee \\ &\simeq \det\Big(Rp_*\big(\mathcal{F} \otimes [V]\big)\Big)^\vee\end{aligned}$$

where $[V]$ is any numerical K-theory class of $\mathrm{ch}(V') = (2, 2(g-1)-d)$. We may choose a representative

$$[V] = (\mathcal{O}_X)^{\oplus 2} \oplus k(x)^{\oplus(2(g-1)-d)} \in K^0(X)_{\mathrm{num}}$$

hence obtaining

$$\begin{aligned}\pi^*\Theta &\simeq \det\Big(Rp_*\mathcal{F}\Big)^{\otimes(-2)} \otimes \det\Big(\mathcal{F}\Big|_{P_w \times \{x\}}\Big)^{\otimes(d+2-2g)} \\ &\simeq \mathcal{O}_w(2, -2g+2d) \otimes \mathcal{O}_w(0, -d-2+2g) \\ &\simeq \mathcal{O}_w(2, d-2).\end{aligned}$$

□

**Remark 121.** Since $\pi : P_w \to M(2,L)$ is a projective bundle, we have $\pi_*\mathcal{O}_{P_w} \simeq \mathcal{O}_{M(2,L)}$. Therefore, we can identify the Verlinde space as

$$\begin{aligned}H^0(M(2,L), \Theta^{\otimes k}) &\simeq H^0(P_w, \pi^*\Theta^{\otimes k}) \\ &\simeq H^0(P_w, \mathcal{O}_w(2k, (d-2)k)) \\ &\simeq V_{2k,(d-2)k}\end{aligned}$$

where $k \geq 0$.

13.2. **Ample cone of moduli space of pairs.** It is in general very hard to determine exactly what the ample cone is for a variety. However, this is possible in our case due to GIT construction.

**Theorem 122.** *For $0 < i < w$, the ample cone of $P_i$ is bounded by $\mathcal{O}_i(1, i-1)$ and $\mathcal{O}_i(1, i)$. The ample cone of $P_w$ is bounded by $\mathcal{O}_w(1, w-1)$ and $\mathcal{O}_w(2, d-2)$.*

**Remark 123.** Since $\mathbb{Z} \oplus \mathbb{Z} \simeq \mathsf{Pic}(P_i)$ for $i = 1, \ldots, w$, we may describe ample cone of each $P_i$ in a fixed plane (over the rational number)

$$\mathbb{Q} \oplus \mathbb{Q} := \{(m, n) \,|\, m,\ n \in \mathbb{Q}\}.$$

Above theorem says that ample cone "flips" as we wall cross from $P_{i-1}$ to $P_i$.

*Proof.* First part of the proof is to show that line bundles in the prescribed region above are indeed ample. Second part of the proof is to show that boundaries of those line bundles are not ample by intersecting with $\mathbb{P}(W_i^{\pm})$.

For the first part, we use the ample line bundles that come from GIT construction. We briefly recall GIT construction of $P_i$. First fix any stability parameter $\sigma \in (d - 2(i+1), d - 2i) \cap \mathbb{Q}_{>0}$ that yields the moduli space $P_i$. We have $\mathrm{SL}(V)$-equivariant morphism

$$f \times 1 : U \times \mathbb{P}(V) \to \mathbb{P}\big(\mathrm{Hom}(\Lambda^2 V, H^0(L)\big) \times \mathbb{P}(V)$$

where $U \subset \mathsf{Quot}_X(V \otimes \mathcal{O}_X, (2, d))$ is some locally closed subset. Consider $\mathrm{SL}(V)$-linearized ample line bundle

$$\mathcal{L}_\sigma := \mathcal{O}(\chi + \sigma, 2\sigma)$$

on $\mathbb{P}\big(\mathrm{Hom}(\Lambda^2 V, H^0(L)\big) \times \mathbb{P}(V)$. This defines a GIT semistable locus $V'(\sigma)$ with respect to $\mathcal{L}_\sigma$. Let $V(\sigma) := (f \times 1)^{-1}(V'(\sigma))$. Then finite morphism

$$(f \times 1) : V(\sigma) \to V'(\sigma)$$

induces a finite morphism between good quotients

$$P_i = V(\sigma)/\mathrm{SL}(V) \to V'(\sigma)/\mathrm{SL}(V).$$

On the other hand, $V'(\sigma)/\mathrm{SL}(V)$ is equipped with an ample line bundle descent from $\mathcal{L}_\sigma$ as a GIT quotient. Since $P_i \to V'(\sigma)/\mathrm{SL}(V)$ is finite morphism, the pullback of this line bundle also gives an ample line bundle. First part of the claim then follows once we prove that this pullback line bundle is $\mathcal{O}_i(1, d-1-\sigma) \in \mathsf{Pic}(P_i)$. This can be done by descent theory which we omit in this lecture note.

For the second part, one can show that restriction of $\mathcal{O}_i(m, n)$ to the fiber projective space of $\mathbb{P}(W_i^+) \subset P_i$ is $\mathcal{O}(n - (i-1)m)$. Similarly, one can show that restriction of $\mathcal{O}_{i-1}(m, n)$ to the fiber projective space of $\mathbb{P}(W_i^-) \subset P_{i-1}$ is $\mathcal{O}((i-1)m - n)$. Proof of these restriction computation will appear in the next section. For a line bundle $\mathcal{O}_i(m,n)$ to be ample, we need to have positivity for the restriction to the projective fiber. This excludes the boundary cases. $\square$

Using the knowledge of the ample cone of each $P_i$ and the formula of canonical bundle $K_{P_i} = \mathcal{O}_i(-3, 4-d-g)$, we understand when is

$$K_{P_i}^{\vee} \otimes \mathcal{O}_i(m,n)$$

is in the ample cone.

**Corollary 124.** *Suppose that $m, n \geq 0$ and that $m(d-2) - 2n > -d + 2g - 2$. Then we have*

$$\dim(V_{m,n}) = \chi(P_b, \mathcal{O}_b(m,n))$$

*for* $b = \left[\frac{n+d+g-4}{m+3}\right] + 1$.

**Remark 125.** Condition $m(d-2) - 2n > -d + 2g - 2$ in the above statement guarantees that $1 < b < w$ hence the moduli space $P_b$ exists. In the proof of the above corollary, Kodaira vanishing on $M_b$ is not sufficient when the corresponding line bundle is in the boundary of the ample cone. However, this case is also treated in the paper of Thaddeus.

13.3. **Wall-crossing formula for Riemann-Roch numbers.** Throughout this section, we always assume the conditions

$$m, n \geq 0, \quad m(d-2) - 2n > -d + 2g - 2$$

as in Corollary 124 and let

$$b = \left[\frac{n+d+g-4}{m+3}\right] + 1.$$

By convention, we let $\mathcal{O}_0(m,n) := \mathcal{O}(m+n)$ on $P_0 \simeq \mathbb{P}^{d+g-2}$. Consider the formula

$$\dim(V_{m,n}) = \sum_{i=0}^{b} \Big( \chi(P_i, \mathcal{O}_i(m,n)) - \chi(P_{i-1}, \mathcal{O}_{i-1}(m,n)) \Big)$$

where we recall that $P_{-1} = \emptyset$ hence $\chi(P_{-1}, *) = 0$. Therefore it suffices to compute the wall-crossing terms in each summand. Since geometry of wall-crossing at each step involves a symmetric product $\mathrm{Sym}^i(X)$ and two bundles

$$W_i^- = p_*\big(\mathcal{O}_{\mathcal{D}}(-\mathcal{D}) \otimes q^*L\big), \quad W_i^+ = \mathcal{E}\mathrm{xt}^1_p\Big(q^*L(-\mathcal{D}), \mathcal{O}(\mathcal{D})\Big),$$

it is natural to express each wall-crossing term using these data. We additionally need a line bundle

$$L_i := \det\Big(Rp_*L(-\mathcal{D})\Big)^{\vee} \otimes \det\Big(Rp_*\mathcal{O}(\mathcal{D})\Big)^{\vee}$$

and an integer $q_i := n - (i-1)m$. Define

$$N_i := \chi\Big(\mathrm{Sym}^i(X)\,,\, L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_i - i} U_i\Big)$$

where $U_i := W_i^- \oplus (W_i^+)^\vee$. By convention, we let $N_i = 0$ if $q_i - i < 0$. Note that the formula of $N_i$ makes sense not just for $0 \le i \le b$ but for every $i \ge 0$.

**Proposition 126.** *Under the assumption of Corollary 124, we have*

(1) $\chi(P_0, \mathcal{O}_0(m,n)) = N_0$,
(2) $N_i = 0$ *for* $i > b$,
(3) $\chi(P_i, \mathcal{O}_i(m,n)) - \chi(P_{i-1}, \mathcal{O}_{i-1}(m,n)) = (-1)^i N_i$ *for* $0 < i \le b$.

*In particular,*

$$\dim(V_{m,n}) = \sum_{i=0}^{\infty} (-1)^i N_i.$$

*Proof.* For the first statement, note that $\mathrm{Sym}^0(X)$ is a point hence

$$N_0 = \mathrm{rk}\Big(L_0^m \otimes \Lambda^0 W_i^- \otimes S^{q_0 - 0} U_0\Big) = \mathrm{rk}\Big(S^{n+m}\big(W_0^- \oplus (W_0^+)^\vee\big)\Big).$$

Since $W_0^- = 0$, $W_0^+ = \mathrm{Ext}^1(L, \mathcal{O})$ and $P_0 = \mathbb{P}(W_0^+)$, it is clear that $N_0 = \chi(P_0, \mathcal{O}_0(m,n))$. The second statement follows by arithmetic that $i > b = \left[\frac{n+d+g-4}{m+3}\right] + 1$ implies $q_i - i < 0$ hence $N_i = 0$.

The third statement is at the heart of the proposition. We recall and introduce some key facts about the blow up diagram

$$\begin{tikzcd}
 & & E_i \arrow[dll] \arrow[drr] \arrow[d, hook] & & \\
 & & \widetilde{P}_i \arrow[dl, "f^+"'] \arrow[dr, "f^-"] & & \\
\mathbb{P}(W_i^+) \arrow[r, hook] \arrow[ddrr, "F^+"'] & P_i \arrow[dr, "\pi_+"'] & & P_{i-1} \arrow[dl, "\pi_-"] & \mathbb{P}(W_i^-) \arrow[l, hook'] \arrow[ddll, "F^-"] \\
 & & P_X^{(d-2i)-ss}(2,L) & & \\
 & & \mathrm{Sym}^i(X). \arrow[u, hook] & &
\end{tikzcd}$$

Since $f^\pm$ is a blow up morphism over the smooth center, we have

$$Rf^+_* \mathcal{O}_{\widetilde{P}_i} = \mathcal{O}_{P_i}, \quad Rf^-_* \mathcal{O}_{\widetilde{P}_i} = \mathcal{O}_{P_{i-1}}.$$

Therefore

$$\chi(P_i, \mathcal{O}_i(m,n)) - \chi(P_{i-1}, \mathcal{O}_{i-1}(m,n)) = \chi(\widetilde{P}_i, \mathcal{O}_i(m,n)) - \chi(\widetilde{P}_i, \mathcal{O}_{i-1}(m,n))$$

where we omitted the pullback notation for line bundles. On the other hand, line bundles $\mathcal{O}_i(m,n)$ and $\mathcal{O}_{i-1}(m,n)$ agree on $\widetilde{P}_i$ away from $E_i = \mathbb{P}(W_i^-) \times_{\mathrm{Sym}^i(X)} \mathbb{P}(W_i^+)$ hence there is some $k \in \mathbb{Z}$

such that

$$\mathcal{O}_i(m,n) = \mathcal{O}_{i-1}(m,n)(kE_i).$$

But $\mathcal{O}_i(m,n)$ on $\widetilde{P}_i$ must be trivial on fibers of $\mathbb{P}(W_i^-)$ in $E_i$. This can be used to determine that

$$k = (i-1)m - n = -q_i.$$

To compare $\chi(\widetilde{P}_i, \mathcal{O}_{i-1}(m,n)(-q_iE_i))$ and $\chi(\widetilde{P}_i, \mathcal{O}_{i-1}(m,n))$, we use the exact sequence

$$0 \to \mathcal{O}(-E_i) \to \mathcal{O}_{\widetilde{P}_i} \to \mathcal{O}_{E_i} \to 0$$

and the fact that $\mathcal{O}_{i-1}(m,n)$ restricted to $E_i$ is $L_i^m(-q_i, 0)$. We divide the cases when $-q_i \geq 0$ or $q_i > 0$.

We first assume that $-q_i \geq 0$, in which case $q_i - i < 0$ hence $N_i = 0$. For each $0 < j \leq -q_i$, we have an exact sequence

$$0 \to \mathcal{O}_{i-1}(m,n)((j-1)E_i) \to \mathcal{O}_{i-1}(m,n)(jE_i) \to L_i^m(-q_i - j, -j) \to 0.$$

Therefore we obtain

$$\chi(\widetilde{P}_i, \mathcal{O}_i(m,n)) - \chi(\widetilde{P}_i, \mathcal{O}_{i-1}(m,n)) = \sum_{j=1}^{-q_i} \chi(E_i, L_i^m(-q_i - j, -j)).$$

On the other hand, condition in Corollary 124 implies that $0 < j < d + g - 1 - 2i$ for all $j$ in the summation above. This implies the desired vanishing of the right hand side since $\mathrm{rk}(W_i^+) = d + g - 1 - 2i$.[26]

Now we assume that $q_i > 0$. By the same method above we have

$$\chi(\widetilde{P}_i, \mathcal{O}_i(m,n)) - \chi(\widetilde{P}_i, \mathcal{O}_{i-1}(m,n)) = -\sum_{j=0}^{q_i - 1} \chi(E_i, L_i^m(-q_i + j, j)).$$

We pushforward the computation of each term on the right hand side to $\mathrm{Sym}^i(X)$. Since each fiber of $E_i \xrightarrow{F} \mathrm{Sym}^i(X)$ is of the form $\mathbb{P}^{i-1} \times \mathbb{P}^{d+g-2-2i}$ and $-q_i + j < 0 \leq j$, every higher direct image sheaf vanish other than potentially non-trivial

$$\begin{aligned} R^{i-1}F_*(L_i^m(-q_i + j, j)) &= L_i^m \otimes R^{i-1}F_*^-(\mathcal{O}_{\mathbb{P}(W_i^-)}(-q_i + j)) \otimes R^0F_*^+(\mathcal{O}_{\mathbb{P}(W_i^+)}(j)) \\ &= L_i^m \otimes \Lambda^i W_i^- \otimes \left(R^0F_*^-(\mathcal{O}_{\mathbb{P}(W_i^-)}(q_i + j - i))\right)^\vee \otimes R^0F_*^+(\mathcal{O}_{\mathbb{P}(W_i^+)}(j)) \\ &= L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_i - j - i}(W_i^-) \otimes S^j(W_i^+)^\vee. \end{aligned}$$

We used a relative Serre duality in the second equality. This term clearly vanishes unless the index $0 \leq j < q_i$ satisfies $q_i - j - i \geq 0$. On the other hand,

$$\bigoplus_{j=0}^{q_i - i} S^{q_i - j - i}(W_i^-) \otimes S^j(W_i^+)^\vee = S^{q_i - i}(W_i^- \oplus (W_i^+)^\vee).$$

[26]We are using a family version of the vanishing $\chi(\mathbb{P}^n, \mathcal{O}(-m)) = 0$ for $0 < m \leq n$.

Therefore we prove that

$$\chi(\widetilde{P}_i, \mathcal{O}_i(m,n)) - \chi(\widetilde{P}_i, \mathcal{O}_{i-1}(m,n)) = (-1)^i \chi\big(\mathrm{Sym}^i(X), L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_i - i} U_i\big).$$

□

13.4. **Intersection theory on the symmetric product of curves.** To finish the computation, we study intersection theory on the symmetric product $\mathrm{Sym}^i(X)$. Goal of this section is to compute the Riemann-Roch number

$$N_i = \chi(\mathrm{Sym}^i(X), L_i^m \otimes \Lambda^i W_i^- \otimes S^{q_i - i} U_i)$$

that appears as a Wall-crossing term.

Let $1_X \in H^0(X, \mathbb{Z})$ be a fundamental class and $[\mathrm{pt}] \in H^2(X, \mathbb{Z})$ be a Poincare dual for the point class. Choose a symplectic basis $\{e_j, f_j\}_{1 \le j \le g}$ of $H^1(X, \mathbb{Z})$ satisfying $\langle e_j, f_j \rangle = -\langle f_j, e_i \rangle = 1$. Using Kunneth decomposition of the divisor class $[\mathcal{D}]$, we define $\eta \in H^2(\mathrm{Sym}^i(X), \mathbb{Z})$ and $\zeta_j, \zeta_j'$ satisfying

$$[\mathcal{D}] = \eta \otimes 1_X + \sum_{j=1}^{g} (\zeta_j' \otimes e_j - \zeta_j \otimes f_j) + i \cdot 1_{\mathrm{Sym}^i(X)} \otimes [\mathrm{pt}] \in H^2(\mathrm{Sym}^i(X) \times X, \mathbb{Z}).$$

It is known that $\zeta_j, \zeta_j', \eta$ generate the cohomology ring $H^*(\mathrm{Sym}^i(X), \mathbb{Z})$. Furthermore, if we put $\sigma_j := \zeta_j \zeta_j' \in H^2(\mathrm{Sym}^2(X), \mathbb{Z})$, then $\sigma_j^2 = 0$ and

$$\int_{\mathrm{Sym}^i(X)} \eta^{i-|I|} \cup \sigma_I = 1$$

where $I \subseteq \{1, \ldots, g\}$. Put $\sigma := \sum_{j=1}^{g} \sigma_j$. Then above formula implies that for any power series $A(x), B(x)$ we have

$$\begin{aligned}
\int_{\mathrm{Sym}^i(X)} A(\eta) \exp(B(\eta)\sigma) &= \sum_{k=0}^{\infty} \int_{\mathrm{Sym}^i(X)} A(\eta) B(\eta)^k / k! \\
&= \sum_{k=0}^{g} \int_{\mathrm{Sym}^i(X)} \binom{g}{k} \mathrm{Res}_{\eta=0} \left( \frac{A(\eta) B(\eta)^k}{\eta^{i-k+1}} d\eta \right) \\
&= \mathrm{Res}_{\eta=0} \left( \frac{A(\eta)(1 + \eta \cdot B(\eta))^k}{\eta^{i+1}} d\eta \right).
\end{aligned}$$

For the computation of Riemann-Roch number, we need a todd class formula

$$\mathrm{td}(\mathrm{Sym}^i(X)) = \left( \frac{\eta}{1 - e^{-\eta}} \right)^{i-g+1} \exp\left( \frac{\sigma}{e^{\eta} - 1} - \frac{\sigma}{\eta} \right)$$

which can be obtained from the tangent bundle formula

$$T_{\mathrm{Sym}^i(X)} = p_* \mathcal{O}_{\mathcal{D}}(\mathcal{D}).$$

By Grothendieck-Riemann-Roch formula we can also obtain the chern character of the input in the definition of $N_i$. More precisely, we have

(1) $\mathrm{ch}(L_i) = \exp((d-2i)\eta + 2\sigma)$,
(2) $\mathrm{ch}(\Lambda^i W_i^-) = \exp((d-3i+1-g)\eta + 3\sigma)$,
(3) $\mathrm{ch}(U_i) = (d-i+1-2g)e^{-\eta} + (2g-2)e^{-2\eta} + \sum_{j=1}^{g} e^{-\eta-\sigma_j}$.

Using the chern character formula of the symmetric product $S^{q_i-i}U_i$, we can express the Riemann-Roch number as

$$N_i = [t^{q_i-i}]\,\mathrm{Res}_{\eta=0}\left(\frac{e^{((d-2)m-2n)\eta}\cdot(e^{-\eta}-t)^{-d+i-1+g}}{(1+t)^{2g-2}(1-e^{-\eta})^{i+1}}\cdot\left(e^{-\eta}+(2m+3-\tfrac{t}{e^{-\eta}-t})(1-e^{-\eta})\right)^g d\eta\right)$$

after careful computations. By the change of variable

$$y = \frac{e^{-\eta}-t}{1-e^{-\eta}},$$

we can rewrite the residue as

$$N_i = [t^{q_i-i}]\,\mathrm{Res}_{y=0}\left(\frac{a(y)}{y^{i+1}}dy\right)$$

where

$$a(y) := \frac{(1+ty)^{2q_{d/2}-1}(1+y)^{-2q_{d/2}+d-2g+1}}{(1-t)^{d+g-1}}(1+(2m+3)(1-t)y-ty^2)^g.$$

Therefore we obtain

$$\begin{aligned}\dim(V_{m,n}) &= \sum_{i=0}^{\infty}(-1)^i N_i\\ &= \sum_{i=0}^{\infty}(-1)^i[t^{(m+n)-(m+1)i}][y^i]a(y)\\ &= [t^{m+n}]\sum_{i=0}^{\infty}(-t^{m+1})^i[y^i]a(y)\\ &= [t^{m+n}]a(-t^{m+1}).\end{aligned}$$

Define

$$\begin{aligned}F(t) :&= \frac{a(-t^{m+1})}{t^{m+n}}\\ &= \frac{(1-t^{m+2})^{-h-1}(1-t^{m+1})^{-h'-1}}{(1-t)^{d+g-1}t^{m+n}}(1-(2m+3)(1-t)t^{m+1}-t^{2m+3})^g,\end{aligned}$$

where $h = (d-2)m-2n$ and $h' = -h-d+2g-2$. In sum up, we have the following theorem.

**Theorem 127.** *Under the assumption of Corollary 124, we have the formula*

$$\dim(V_{m,n}) = \mathrm{Res}_{t=0}\left(\frac{F(t)}{t}dt\right).$$

By computation of order of poles and zeros of $F(t)/t$, one can show that it has no pole at $t=1$. Therefore, there are residues of four possible kinds: $t=0$, $t=\infty$, $t^{m+1}=0$ with $t\neq 1$ and

$t^{m+2}=1$ with $t\neq 1$. One the other hand, it is easy to check that $F(1/t)=-F(t)$ which implies that

$$\mathrm{Res}_{t=0}\left(\frac{F(t)}{t}dt\right)=\mathrm{Res}_{t=\infty}\left(\frac{F(t)}{t}dt\right).$$

By residue theorem, sum of all residues must be zero. Therefore, we obtain the formula

$$\dim(V_{m,n})=-\frac{1}{2}\left(\sum_{\substack{\zeta^{m+1}=1\\ \zeta\neq 1}}\mathrm{Res}_{t=\zeta}+\sum_{\substack{\zeta^{m+2}=1\\ \zeta\neq 1}}\mathrm{Res}_{t=\zeta}\right)\left(\frac{F(t)}{t}dt\right). \tag{6}$$

Computation of this can be demanding in general but it simplifies for the Verlinde case as we see below.

Recall that $V_{m,n}$ specializes to Verlinde vector space

$$V_{2k,(d-2)k}=H^0(M_X(2,L),\Theta^{\otimes k}),\quad k\geq 0.$$

In such a case, assumption of Corollary 124 is equivalent to simply $d>2g-2$ which trivial as we assume that $d$ is sufficiently large compared to $g$. In the Verlinde case, formula of $F(t)$ simplifies because $h=0$ and $h'<0$. In this case, it suffices to consider residues at $t=\zeta$ such that $\zeta^{m+2}=1$ with $\zeta\neq 1$. The residue formula reads

$$\dim(V_{2k,(d-2)k})=-\frac{1}{2}\sum_{\substack{\zeta^{2k+2}=1\\ \zeta\neq 1}}\mathrm{Res}_{t=\zeta}\left(\frac{-dt/t}{t^{2k+2}-1}\right)\cdot\frac{(1-\zeta^{-1})^{d-2g+1}}{(1-\zeta)^{d+g-1}\zeta^{kd}}\cdot(1-(4k+3)(\zeta^{-1}-1)-\zeta^{-1})^{g}.$$

By computation, one obtain

$$\dim(V_{2k,(d-2)k})=(4k+4)^{g-1}\sum_{\substack{\zeta^{2k+2}=1\\ \zeta\neq 1}}\frac{(-1)^{d+g-1}\zeta^{(k+1)d}}{(\zeta^{-1}-\zeta)^{2g-2}}.$$

This is equivalent to the Verlinde formula below.

**Theorem 128.** *Let $X$ be a smooth projective curve of genus $g\geq 2$. Let $M_X(2,L)$ be a moduli of stable bundles of rank 2 and odd determinant $L$. For a level $k$ theta line bundle $\Theta^{\otimes k}$, we have*

$$\dim\, H^0(M_X(2,L),\Theta^{\otimes k})=(k+1)^{g-1}\sum_{j=1}^{2k+1}\frac{(-1)^{j-1}}{\left(\sin\frac{j\pi}{2k+2}\right)^{2g-2}},\quad k\geq 0.$$

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